MATH 1500, Homework 10

Due date: Wed, 13 January

1. Evaluate the following integrals (note that parts (d,f,j) are definite integrals; the rest are indefinite).

$$(a)\ \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx \quad (b)\ \int \frac{\ln t}{t}dt \quad (c)\ \int_0^1 \frac{2}{3+4x^2}dx; \quad (d)\ \int \frac{x^2}{2+x^6}dx$$

$$(e)\ \int_0^\pi \cos^4 x dx \quad (f)\ \int \sqrt{\tan x}\sec^4 x dx \quad (g)\ \int_0^1 \arctan x \quad (h)\ \int_0^1 \frac{1}{1+x^{1/3}}dx$$

2. A left-point rule is an approximation to $\int_a^b f(x)dx$ using Reimann sum with each rectangle taken to be at the left endpoint. That is,

$$L_n = \sum_{i=1}^n f(x_{i-1})\Delta x$$

where $x_i = a + \frac{(b-a)}{n}i, \quad i = 0 \dots n$ and $\Delta x = \frac{b-a}{n}$. The right-point and midpoint rules are defined analogously.

   (a) Estimate $\int_0^1 x^2 dx$ using the left-point, midpoint and right-point rule with two subintervals.

   (b) Show that if $f'(x) > 0$ then the leftpoint rule is an underestimate of $\int_a^b f(x)dx$ and the rightpoint rule is its overestimate.

   (c) [BONUS] Show that if $f'(x) > 0$ and $f''(x) > 0$ for $x \in [a, b]$ then the midpoint rule is an underestimate of $\int_a^b f(x)dx$.

3. Recall that we defined

$$\cosh(x) = \frac{e^x + e^{-x}}{2}; \ \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

and we have identities

$$\cosh^2(x) - \sinh^2(x) = 1$$
$$\frac{d}{dx}\cos(x) = \sinh(x); \quad \frac{d}{dx}\sinh(x) = \cosh(x);$$

Similarly, we define

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

   (a) We have the related identities

$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$
$$\frac{d}{dx}\operatorname{sech}(x) = -\tanh(x)\operatorname{sech}(x); \quad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x);$$

   (b) Show that

$$\int_0^\infty \operatorname{sech}(x)\,dx = \frac{\pi}{2}; \quad \int_0^\infty \operatorname{sech}^2(x)\,dx = 1.$$

   (c) Define $I_n = \int_0^\infty \operatorname{sech}^n(x)\,dx$. Show that

$$I_n = \frac{n-2}{n-1}I_{n-2}.$$

(d) Show that $I_1 \leq I_2 \leq I_3 \leq ...$ Hint: first you need to show that $\operatorname{sech}(x) \leq 1$ for all $x \geq 0$.

(e) [BONUS] Using parts (b),(c) and (d), show that for any positive integer n, we have

$$\left(\frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right)^2 \frac{1}{(2n+3)} \leq \frac{\pi}{2} \leq \left(\frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right)^2 \frac{1}{(2n+2)}.$$

(f) [BONUS] Show the *Wallis product formula,*

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7} \cdots = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$