MATH 1500, Homework 7

Due date: 11 November (Wednesday)

1. On the same graph, sketch the functions $e^x, e^{2x}, e^{-x}, e^{-2x}$.

2. For each of the functions below, find their derivatives. Then sketch their graphs. Indicate (if any) roots, max/min and behaviour at endpoints or infinities.

   (a) $f(x) = x\exp(-x), \quad -\infty < x < \infty$

   (b) $f(x) = e^{-(x^2)}, \quad -\infty < x < \infty$

   (c) $f(x) = x\ln(x), \quad 0 < x < \infty$ (make sure to indicate $\lim_{x\to 0^+} f(x)$ as well as behaviour at $+\infty$)

   (d) $f(x) = x^{(1/x)}, \quad 0 < x < \infty$ (make sure to indicate $\lim_{x\to 0^+} f(x)$ as well as behaviour at $+\infty$)

   (a) The function $y = y(x)$ is defined implicitly by the equation $x^2y + x\tan(y) - \pi/4 = 1$. Determine $\frac{dy}{dx}$ at the point $x = 1, y = \pi/4$.

   (b) You are given a function $f(x)$ that satisfies

   $$\frac{d}{dx}[f(x)] = \frac{1}{x^3+1}, \quad f(1) = 2.$$

   Let g be the inverse of f; that is, $f(g(x)) = x$. Determine $g(2)$ and $g'(2)$.

3. The hyperbolic trigonomentric functions are defined as:

   $$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x}$$

   (a) Verify the following identities:

   $$\frac{d}{dx}\cosh x = \sinh x; \quad \frac{d}{dx}\sinh x = \cosh x;$$
   $$\cosh^2 x - \sinh^2 x = 1.$$
   $$\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$$

   (b) Derive an addition formula for $\cosh(x+y)$ which is similar to the addition formula for $\cos$.

   (c) Sketch the graphs of $\sinh x, \cosh x$ and $\tanh x$. Indicate any odd or even symmetry.

4. Recall from class that the solution to the differential equation

   $$\frac{dy}{dt} = F(y)$$

   is given implicitly by

   $$\int \frac{dy}{F(y)} = \int dt + C.$$

   The goal of this question is to derive the solution to the logistic equation,

   $$\frac{dy}{dt} = y(1-y). \tag{1}$$

   (a) Determine constants A, B such that

   $$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{y-1}.$$

(b) Find $\int \frac{1}{y(1-y)} dy$. Hint: use part (a).

(c) Solve (1). If $y(0) = 1/2$, find $y(1)$. [Double-check that $y(1) = 0.731$; but you are asked to find it explicitly in terms of "e".]

5. A marble rolls along a straight line in such a manner that its velocity is directly proportional to the distance it has *yet* to roll. If the total distance it rolls is 5 meters, and after 1 second it has rolled 2 meters, express the distance it has rolled as a function of time. How long will it take for the marble to roll 4 meters?

6. Newton's law of cooling states the rate of change of temperature of a small object is proportional to the the difference between its temperature and that of the surrounding environment. That is,

$$\frac{dT}{dt} = k\,(T - T_e)$$

where T and T_e is the is the temperature of the object and T_e is the temperature of the surrounding environment. Apply Newton's law to answer the following question.

(a) When a cup of tea is made, its temperature is initially 90^o. A minute later, its temperature has cooled to 70^o. If the temperature outside is 20^o, how long will it take for the temperature of the cup to reach 30^o?

7. (a) Sketch the graph of $f(\theta) = \tan(\theta)$.

(b) Make an appropriate restriction on the domain of $\tan\theta$ and define $\arctan x$. What is the domain and range of $\arctan x$?

(c) Sketch the graph of $\arctan x$ and show that

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

(d) What is $\arctan(1)$?

8. [BONUS] A roll of toilet paper is placed on the floor against the wall. A crayon is fixed at the centre of the roll. As the roll is being unrolled, the crayon leaves a trace on the wall. What is the shape of the curve drawn by the crayon?