MATH 1500, Homework 8

Due date: Monday 23 November.

1. Section 4.9: 17, 18, 27.

2. A given function $F(x)$ satisfies:

$$F(1) = 1; \quad F'(1) = 2, \; F''(1) = 3, \; F'''(x) = 4x - 5\sin\left(x^6\right) + \frac{7}{x}.$$

   (a) Estimate $F(0.9)$ using quadratic approximation. What is the maximum error in your estimate? Hint: recall that $|a+b| \leq |a| + |b|$.

3. Write the full Taylor series of the following functions, centered at $x = 0$. Write down enough terms so that the general pattern is visible.

$$\begin{aligned}
(a) \quad & f(x) = \sin(x^2) \\
(b) \quad & f(x) = \frac{1}{1+x} \\
(c) \quad & f(x) = \frac{1}{1+x^2} \\
(d) \quad & f(x) = \frac{1}{2-x} \\
(e) \quad & f(x) = \frac{1}{2+x^2} \\
(f) \quad & f(x) = \ln(2+x^2)
\end{aligned}$$

   HINT: All you need is the Taylor series for $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and for $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

   HINT2: For (d), write $\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-x/2}\right)$.

4. (a) Note that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$. Use this fact to show that the full Taylor series for $f(x) = \frac{1}{(1-x)^2}$ is

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

   (b) Note that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$. Use this fact and question (3c) to determine the full Taylor series for $f(x) = \arctan x$.

   (c) Recall that $\tan(\pi/6) = \frac{1}{\sqrt{3}}$. Use this fact and your answer in (b) to prove that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3}\frac{1}{3} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\frac{1}{3^3} + \frac{1}{9}\frac{1}{3^4}\cdots\right)$$

   Use this series to determine π to three decimal places. How many terms did you need?

5. Section 4.3 numbers 1, 2, 3, 13, 16. [You can use either using Taylor series or L'Hopital's rule].