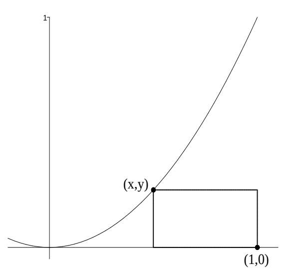
Review Questions for Midterm 2

1. (a) A curve is implicitly given by the equation
$$y^3 - y + x + x^3 = 8.$$
Find $\frac{dy}{dx}$ at the point $x = 1, y = 2$.
   (b) Find the derivative of $y = x^{1/x}$. Sketch its graph for $x > 0$.
   (c) Define arcsec. What is its domain and range? Find its derivative. [Recall: sec=1/cos].
   (d) Find the derivative of $y = (\sin x)^{\cos x}$.

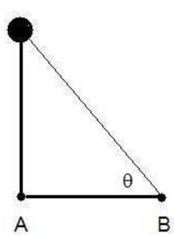
2. Sketch the graphs of given functions. Include any max/min, roots, asymptotes.
$$\begin{aligned} (a) \quad f(x) &= x - \frac{1}{x^2} \\ (b) \quad f(x) &= e^{-1/x} \\ (c) \quad f(x) &= \frac{1}{1+e^{-x}} \\ (d) \quad f(x) &= \frac{\ln x}{x^2} \end{aligned}$$

3. One of the vertices of a rectangle has coordinates (1,0). The opposite vertex (x, y) lies on the parabola $y = x^2$. How should (x, y) be chosen to maximize the area of such a rectangle, if $x \leq 0 \leq 1$? Assume all of the sides of the rectangle are parallel to the x and y axes.



4. (a) Find the dimensions of the largest rectangle that can be inscribed in a circle of radius R.
   (b) Find the dimensions of the largest cylinder that can be inscribed in a sphere of radius R.

5. A balloon released at point A rises vertically with a constant speed of 4 m/s. An observer at point B is level with and 100 m distant from point A. How fast is the angle of elevation (θ) of the balloon, as seen from B, changing when the balloon is 200m above A?



6. A corpse was discovered in a motel room and its temperature was 26°C. The temperature of the room is kept constant at 15°C. Two hours later the temperature of the corpse dropped to 24°C. The normal human temperature is 37°C. Assuming the person was healthy at the time of murder, how was he dead before he was found? Note: assume Newton's law of cooling, which states that the rate of change of temperature of a body is proportional to the difference between its temperature and that of the surrounding environment.

7. Using the fact that $e = 2.718$ and $3 = e + 0.282$, estimate ln(3) using linear and quadratic approximations. In each case, find a bound for the error.

8. Let $F(x)$ be be such that

$$F(0) = 0; \quad F'(x) = e^{-x^2}.$$

[That is, $F(x)$ is the area under the bell-shaped curve e^{-x^2} from 0 to x].

(a) Estimate $F(\frac{1}{2})$ using linear approximation and determine the error bound.

(b) Write down the first three non-zero terms in Taylor series for $F(x)$, centered at $x = 0$. Then estimate $F(\frac{1}{2})$ as accurately as you can.

9. (a) Find the Taylor series expansion of

$$f(t) = (1+t)^{-1/2}$$

around $t = 0$.

(b) Write down the Taylor series for $\arcsin(x)$ around $x = 0$.

(c) Recalling that $\sin\frac{\pi}{6} = 1/2$, use the result in (b) to come up with an infinite series whose sum is π. Then use it to estimate π to four significant digits.

10. Determine the following limits.

$$(a) \lim_{x\to 0} \frac{\tan 3x}{\arcsin 2x}$$

$$(b) \lim_{x\to 0} \frac{e^{x^3} - 1}{x^3}$$

$$(c) \lim_{x\to 0} \frac{\sqrt{1+x^2} - \cos x}{(\sin x)^2}$$

$$(d) \lim_{x\to 0} \frac{\arcsin x - \sin x}{\ln(1-x^3)} \text{ (hint: make use of question 9b)}$$