Midterm review sheet

- Topics included (homework 1 to 4):

    - limits, continuity
    - Definition of the derivative
    - Differentiation rules
    - Related rates questions.

- How to study:

    - Review all the homework questions as well as their solutions
    - Go over the problem sheets (particularly related rates questions) that were handed out in class.
    - Do additional questions below.

- Additional questions:

1. **Note: At least one part of this question will appear on the midterm.**

    (a) State the delta-epsilon definition of a limit.

    (b) Part I: Find δ such that $\left|\frac{1}{x} - \frac{1}{2}\right| \leq 0.1$ whenever $|x - 2| < \delta$. Part II: Use the delta-epsilon definition directly to show that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

    (c) Let $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$. Use the delta-epsilon definition directly to show that $\lim_{x \to 0} f(x)$ does not exist.

2. Find the following limits (the answer may be $+\infty$ or $-\infty$).

    $$\text{(a)} \quad \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \qquad \text{(b)} \quad \lim_{x \to \infty} \frac{x^2 - 2x^3 + \sqrt{3x^4 + 4x^5}}{x^2 + 3x^3} \qquad \text{(c)} \quad \lim_{x \to 1^+} \frac{1}{1 - x^2}$$

3. (a) Write down the definition of derivative of $f(x)$ as a limit. (b) Let $f(x) = \sqrt{x}$. Use the definition of derivative as a limit to directly compute $f'(4)$. (No marks will be given for using differentiation rules)

4. Find the derivatives of the following functions (using differentiation rules).

    $$\text{(a) } f(x) = \sqrt{(x + 3)^2 + 3x} \qquad \text{(b) } f(x) = \frac{1}{1 - x^2} \qquad \text{(c) } f(x) = \sqrt{x^3 + \sin(3x^2)}$$

5. (a) State the intermediate value theorem. (b) Show that the equation $x^3 = \sin x + 1$ has at least one root.

6. (a) State the mean value theorem. (b) Suppose that $f'(x) \geq x$ for all $x \in [0, 1]$ and $f(0) = 0$. Show that $f(1) \geq \frac{1}{2}$.

7. Water is pumped into a cylindrical tank of radius 3 m at a rate of 1m^3 per min. How fast does the water level in the tank rise?

8. A ladder 5 meters long is placed against a wall. The ladder begins to slip. How fast is the top of the ladder sliding down when the bottom of the ladder is 3 meters away from the wall and is moving to the right at 1/3 meters/sec?
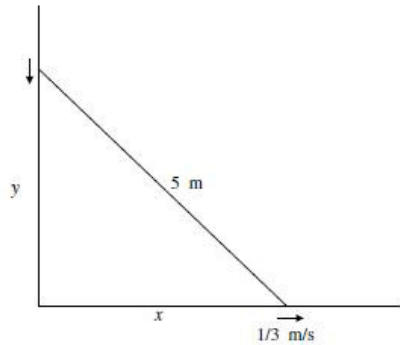
9. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

10. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10cm.

11. At noon, a sailboat is 20 km due south of a freighter. The sailboat is travelling due east at 20 km/hr, and the freighter is travelling due south at 40 km/hr. How fast is the distance between the boats changing one hour later?

12. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute (i.e. 8π radians per minute). How fast is the beam of light moving along the shoreline when it is 1km from P?