Some review questions

1. Evaluate $\int_D e^{x+y} dA$ where D is a region bounded by lines $x = 0, y = 0$ and $y + x = 1$. (answer: 1).

2. Evaluate $\int\int\int_D (x^2 + y^2 + z^2)\, dV$ where D is a region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ with $0 \leq z \leq 1$ and with $x, y \geq 0$.

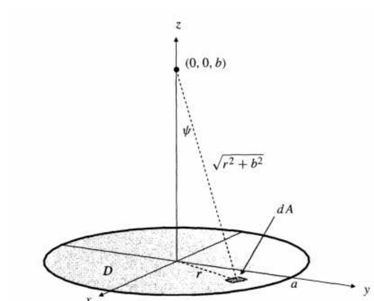
3. A solid ball of radius a has density given by $(2a - \rho)$ where $\rho = \sqrt{x^2 + y^2 + z^2}$. Determine its mass.

4.
   (a) Compute the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. (hint: change the coordinates for the sphere)

   (b) Give the parametrization of the surface of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ in terms of the spherical coordinates.

   (c) Suppose that $a = b$ and $c = 1$. Determine the surface area of the ellipsoid in this case. Answer: $2\pi a \left(a + \frac{\text{arcsinh} \sqrt{|a^2-1|}}{\sqrt{|a^2-1|}} \right)$

5. In this question we compute the graviational attraction F of a point of mass m to a disk of density σ and radius a located a distance b from it and such that the line from the center of the disk to the point mass is perpendicular to the disk itself. Here is the figure:



   Newton's law states that the graviational force between two point masses has the magnitude km_1m_2/d^2 where k is a universal constant.

   (a) Show that $F = km\sigma b \int\int_D \frac{dA}{(r^2+b^2)^{2/3}}$. [hint: horizontal components of the force cancels by symmetry]]

   (b) Evaluate this integral to find that

$$F = 2\pi k m\sigma \left(1 - \frac{b}{\sqrt{a^2 + b^2}} \right).$$

   (c) Deduce that for disks of large radius, the attraction is independent of the distance b.

6. Evaluate $\int\int_S z dS$ where S is a conical surface $z = \sqrt{x^2+y^2}$ between $z=0$ and $z=1$. Answer: $2^{3/2}\pi/3$.

7.

   (a) State Green's theorem.

   (b) Use Green's theorem to prove the divergence theorem in 2D: $\int_D \nabla \cdot \vec{F} dx = \int_{\partial D} \vec{F} \cdot \hat{n} ds$.

8. Show that the center of mass of a simply conneccted region D is given by

$$\bar{x} = \frac{1}{2A}\int_{\partial D} x^2 dy, \quad \bar{y} = \frac{1}{2A}\int_{\partial D} y^2 dx.$$

9. Show the *Green's first identity:*

$$\int\int_D f\Delta g dA = \int_{\partial D} f\nabla g \cdot \hat{n} dS - \int\int_D \nabla f \cdot \nabla g dA$$

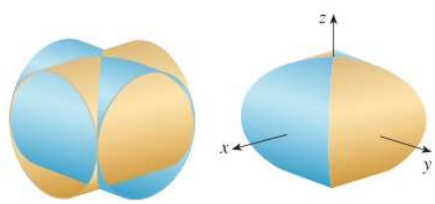
   Here, $\Delta f = \nabla^2 f = \nabla \cdot (\nabla f)$ is the *Laplace operator* [in two dimensions, show that $\Delta f = f_{xx} + f_{yy}$].

10. Show the *Green's second identity:*

$$\int\int_D (f\Delta g - g\Delta f) dA = \int_{\partial D} (f\nabla g - g\nabla f) \cdot \hat{n} dS$$

11. Show that $g(\vec{x}) = \ln|\vec{x}|$, where $\vec{x} = (x, y)$ satisfies $\Delta g = 0$ whenever $\vec{x} \neq 0$.

12. [q62 p.1134] Find area of the surface obtained by intersecting the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.



13. Let S be the surface of the upper half of the unit sphere ($z > 0,\ x^2 + y^2 + z^2 = 1$). Determine the center of mass of this shell.

14. Show the following identities:

$$(a) \quad \mathrm{div}(fF) = f\,\mathrm{div}\,F + F \cdot \nabla f \qquad (b) \quad \mathrm{curl}(fF) = f\,\mathrm{div}\,F + F \cdot \nabla f$$

15. Let $F(\vec{x}) = \vec{x}\,|\vec{x}|^p$, where $\vec{x} = (x, y, z) \in \mathbb{R}^3$. Compute $\mathrm{div}\,F$. Is there a value of p such that $\mathrm{div}\,F = 0$? what about $\mathrm{curl}\,F$?

16. Find the flux of $\vec{F} = (y, z - y, x)$ across the surface of the tetrahedron whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

17. Determine the work done (against gravity) to fill the tetrahedron whose vertices are as in previous question (measured in meters), with water (whose density is 1000 kg/m^3). Assume the water is all on the ground $z = 0$.