MATH 2400, Homework 3

Due: Monday, 10 October

1. Given the data points $(x, y) = (0, 1)$, $(2, 0)$, $(4,\ 2)$, write down the interpolating polynomial through these points using
   (a) Lagrange's polynomial
   (b) Newton's divided differences polynomial.
   (c) Using either (a) or (b), estimate the minimum of a function that goes through these three points (estimate both x and y coordinates of the minimum).

2. Assume that the polynomial $P_9(x)$ interpolates the function $f(x) = \exp(2x)$ at the 10 evenly spaced points $x = 0, 1/9, 2/9, \ldots 8/9, 1$.
   (a) Find an upper bound for the error $|f(1/2) - P_9(1/2)|$. How many decimal places can you guarantee to be correct if $P_9\,(1/2)$ was used to approximate e?
   (b) Why is this a silly way to compute e?

3. List the Chebyshev interpolation nodes $x_1, \ldots x_n$ on the interval $[a, b]$ for
   (a) $n = 5, \quad [a, b] = [-1, 1]$
   (b) $n = 5, \quad [a, b] = [0, 2]$
   (c) $n = 5 \quad [a, b] = [0, 1]$.

   (Note: please only list the first three decimal places)

4. Find $\max_{x\in[a,b]} |(x - x_1) \cdots (x - x_n)|$ with n and $[a, b]$ as given in question 3.

5. Find $\max_{x\in[a,b]} |(x - x_1) \cdots (x - x_n)|$ with n arbitrary and and $[a, b]$ as given in question 3.

6. Let $P_4(x)$, be the 4th degree interpolating polynomial of $\exp(-x)$ using Chebyshev points on the interval $[0, 1]$.
   (a) Using the Interpolation error formula combined with Q5, determine the bound for $|P_4(0.6) - \exp(-0.6)|$.
   (b) Using a computer, plot $P_4(x) - \exp(-x)$. Hand in your plot. How well does it agree with part (a)?

7. The function $y = f(x)$ is tabulated as follows:

$$\begin{array}{ccccc} x_i: & 0 & 1 & 2 & 3 \\ y_i: & 0 & 0 & 1 & 1 \end{array}$$

   (a) Set up a linear system of equations to find the spline through these points with natural endpoint conditions.
   (b) Solve the system you set up in part (a).
   (c) Estimate $f(1.5)$ and $f'(1)$ using the spline approximation you found.
   (d) Using `spline([0,1,2,3], [0,0,1,1], x)` command in Maple or the equivalent command in Matlab/octave, verify your answer to parts (b) and (c).

8. The purpose of this question is to compare the error from Chebychev interpolation vs. uniform interpolation. Let

$$P_n(x) = \prod_{i=1}^{n} \left(x - \frac{i-1}{n-1} \right)$$

   and let $Q_n = \prod_{i=1}^{n} (x - c_i)$ where c_i are the n Chebychev interpolation nodes shifted to the interval [0,1].

(a) Using a computer, plot $P_n(x)$ and $Q_n(x)$ for $n = 10$ and $n = 20$. What can you say about $\max_{x\in[0,1]} |P_n(x)|$ and $\max_{x\in[0,1]} |Q_n(x)|$?

(b) **[BONUS]** Estimate $|P_n(1/2)|$ when n is even and large. How does it compare with $|Q_n(1/2)|$? NOTE: you may find the following formula useful, called the "Sterling's formula":

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n} \quad \text{for large } n.$$

(c) **[BONUS, hard]** What can you say about the $\max_{x\in[0,1]} |P_n(x)|$? How does it compare with $\max_{x\in[0,1]} |Q_n(x)|$?