MATH 2400 HW 4

1. Consider $I = \int_0^1 \exp\left(-x^2\right) dx = 0.7468241328124270$.

   (a) Estimate I using the Trapezoid rule with $n = 4$ subintervals.

   (b) Estimate I using the Simpson's rule with $n = 4$ subintervals.

   (c) How many subintervals n are needed to estimate I with Trapezoid rule to within 10^{-5}?

   (d) Estimate I using Gauss quadrature with $n = 1$ subinterval and $N = 2$ collocation points.

   (e) Estimate I using Gauss quadrature with $n = 1$ subinterval and $N = 3$ collocation points.

   (f) Use Romberg integration to compute I numerically to within 10^{-5} (Use enough iterations until you see that the answer doesnt change anymore in the first 5 digits).

2. Suppose you use a certain numerical method, call it $M(h)$ which depends on the stepsize h to estimate some problem G of great importance. You know that the error has the form $O(h^p)$; that is, $M(h) = G + ah^p + \dots$ as $h \to 0$, but you are not sure what p is. So you make the following table

h	$M(h)$
0.01	0.79818
0.005	0.76642
0.0025	0.74442

   (a) From the data given, estimate p.

   (b) Use the extrapolation along with the answer you obtained in part (a) to estimate G as accurately as you can.

3.

   (a) Consider $I = \int_{-h/2}^{h/2} f(x)$ where $f(x)$ has continuous 2nd derivative. Use the Taylor series expansion of $f(x)$ at $x = 0$ to estimate the error $E = I - hf(0)$.

   (b) Prove the error formula for the midpoint rule: Let

$$M_N = h\left[f(x_1) + f(x_2) \cdots f(x_N)\right]$$
$$\text{where } h = (b-a)/N; \quad x_i = a + hi - h/2$$

   Then $\left|\int_a^b f(x) - M_n\right| \le \frac{M}{24}(b-a)h^2$ where $M \ge \max_{[a,b]} |f''(x)|$.

4. Given that $f(x)$ is sufficiently smooth, find the bound for the error $E = \left|\int_{-1}^1 f(x)dx - M\right|$ where M is one of the following: (a) $M = f(1) + f(-1)$ (b) $M = \frac{2}{3}\left[f(-1) + f(0) + f(1)\right]$ (c) $M = f(-1/\sqrt{3}) + f(1/\sqrt{3})$

5. If a function is smooth and periodic, the Trapezoid rule has a much better convergence. To illustrate this, consider the function

$$f(x) = \exp(\sin(x))$$

   which is periodic with period 2π.

   (a) It is known that $I = \int_0^{2\pi} f(x)dx = 7.95492652101284$, accurate to all digits given. Estimate I using Trapezoid rule T_n with $n = 1, 2, 4, 8$. Make an table of n, T_n, E_n where $E_n = |T_n - I|$ is the error.

   (b) Based on the evidence from part (a), what do you think is the error behaviour?

(c) [**BONUS QUESTION**] Suppose that a function can be expanded in a finite Fourier series as

$$f(\theta) = \sum_{m=-n}^{n} c_m \exp(im\theta).$$

Show that in this case, T_n is an exactly the $\int_0^{2\pi} f(\theta)d\theta$.

(d) [**BONUS QUESTION**] Use part (c) to explain what you observed in parts (a,b).