# MATH 2400 HW 5

Due date: $\leq$11 Nov (Fri)

1. Richardson extrapolation can be used with with ODEs. Here we demonstrate how.

   (a) Let $A(h)$ be the solution to $y'(t) = y(t)$ with $y(0) = 1$ at $t = 1$ using the forward Euler with stepsize $h$; so that $y(1) = A(0)$. In class we have shown that $A(h) - y(1) = O(h)$. In fact, it is possible to show a more precise statement, that

   $$A(h) = A(0) + a_1 h + a_2 h^2 + a_3 h^3 + \dots.$$

   Given $A_1 = A(h)$ and $A_2 = A(h/2)$, show how extrapolation can be used to approximate $y(1)$ up to $O(h^2)$.

   (b) Let $A_1 = A(1),\ A_2 = A(0.5),\ \ A_3 = A(0.25)$ and so on. Print out a table $A_0$ up to $A_6$ (thus $n = 64$ and $h = 1/64$ to compute $A_6$) Knowing that $\exp(1) = 2.7182818284590$, what is the error $A_6 - \exp(1)$?

   (c) Similar to Romberg integration, build a triangular array that approximates $\exp(1)$ as accurately as you can using only the data in part (b). How many digits of $\exp(1)$ did you get? Print out the extrapolation table.

2. Consider the method $y_{i+1} = y_i + h\left(\frac{1}{3}f(y_i) + af(y_i + hbf(y_i))\right)$, as applied to $y' = f(y)$.

   (a) How should you choose $a$ and $b$ to minimize the local error? What is the order of the local and global error of the resulting method?

   (b) Show that the stability region of this method is the same as the stability region for the Trapezoid method $y_{i+1} = y_i + \frac{h}{2}\left(f(y_i) + f(y_i + hf(y_i))\right)$.

3. 

   (a) Implement the variable-step method RK2/3 as described in the book, example 6.19 page 332. Test your RK2/3 on the ODE $y'(t) = y(t)$ with $y(0) = 1$ and with $t \in [0, 1]$. Take the local relative error tolerance to be $T = 0.005$. Output the result, including all the $t_i$ and $y_i$, similar as was done in Figure 6.19 on page 337. What is the minimum and maximum stepsize?

   (b) Repeat part (a), but instead of using relative error tolerance $T\,|w_i| = ch^{p+1}$ as given by formula (6.56) of the book, use the absolute error tolerance given by $T = ch^{p+1}$.

4. Consider the following multi-step method:

   $$y_{i+1} = 3y_i - 2y_{i-1} + h\left(\frac{1}{2}f(y_i) - \frac{3}{2}f(y_{i-1})\right).$$

   (a) Show that this method has $O(h^3)$ local error.

   (b) Use this method to solve $y' = -y$, $y(0) = 1$ with $h = 0.1,\ h = 0.05$. For each $h$, plot the points $(ih, y_i)$ for $i = 0..n$ with $n = 1/h$. Hand in the outputs. What do you observe? NOTE: to plot solution, if using matlab, see p.344 of the book. If using maple, you can use the worksheet `multistep.mws` from the course website and modify it appropriately.

   (c) Show that this method is unstable even if $h = 0$.