1. Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

2. On the sides of a thin rod, heat exchange takes place (obeying Newton's law of cooling – flux proportional to the temperature difference) with a medium of constant temperature T_0. What is the equation satisfied by the temperature $u(x,t)$, neglecting its variation across the rod?

3. Recall the integration by parts formula,
$$\int_D u\ \nabla\cdot\vec{F}\ dx = \int_{\partial D} u\ \vec{F}\cdot\hat{n}\ dS - \int_D \nabla u\cdot\vec{F}\ dx. \tag{1}$$
A special case $u = 1$ yields the Divergence theorem, namely,
$$\int_D \nabla\cdot\vec{F}\ dx = \int_{\partial D} \vec{F}\cdot\hat{n}\ dS. \tag{2}$$
By computing both sides of (2) separately, verify it for the following case:
$$F = r^2(x,y,z),\ \ r = \sqrt{x^2+y^2+z^2}$$
$$D = \text{ball of radius } a \text{ centered at the origin.}$$

4. If $F(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is continuous and $|F(x)| \le 1/\left(|x|^3+1\right)$ for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, use the Divergence theorem to show that
$$\int_{\mathbb{R}^3} \nabla\cdot F dx = 0$$
(hint: integrate over a large ball of radius R and let $R \to \infty$....)

5. Consider the heat equation $u_t = u_{xx}$ with Dirichlet boundary conditions $u(\pm 1, t) = 0$, and an initial condition $u(x,0) = 1 - x^2$.
   (a) Show that the solution $u(x,t)$ is an even function of x, that is $u(-x,t) = u(x,t)$ for any $t \ge 0$ and $x \in (-1,1)$. Hint: make use of uniqueness and invariance of the heat equation under reflections...
   (b) Show that $u(0,t)$ is a decreasing function of t.

6. Solve $u_t = u_{xx}$ on all of $\mathbb{R}$ subject to the initial condition $u(x,0) = x^2$.

7. (a) Solve the PDE $u_t = u_{xx} - cu_x$ subject to initial condition $u(x,0) = \delta(x)$. HINT: Do a change of variables $u(x,t) = U(X,t)$ where $X = x - ct$.

   (b) Solve the problem $u_t = u_{xx} - cu_x$ subject to initial condition $u(x,0) = f(x)$ where $f(x)$ is arbitrary function.

8. Solve $u_t = u_{xx}$ for $x > 0$ subject to the initial condition $u(x,0) = x^2$ and a boundary condition $u(0,t) = 0$.

9. The purpose of this question is to solve $u_t = u_{xx}$ subject to initial condition $u(x,0) = x$ for $x > 0$ and a boundary condition $u_x - 2u = 0$ at $x = 0$.
   (a) Define the function
   $$f(x) = \begin{cases} x, & x > 0 \\ x + 1 - e^{2x}, & x < 0 \end{cases}.$$
   Verify that $f' - 2f$ is an odd function. Then show that $f(x)$ is the unique function such that $f(x) = x$ for $x > 0$, f is continuous, and $f' - 2f$ is odd (for $x \neq 0$).
   (b) Let v solve $v_t = v_{xx}$ subject to initial condition $v(x,0) = f(x)$ for $x \in \mathbb{R}$. Prove that $v(x,t) = u(x,t)$ whenever $x > 0$. Hint: Define $w = v_x - 2v$ and show that $w(0,t) = 0$.
   (c) Write out the solution for u, in terms of certain integrals.