MATH 5250/4250, Homework 4

Due: Thursday, 30 October

1. [Holmes, 2.2 #1] Consider the problem

$$\varepsilon u_{xx} = a - u_x, \quad u(0) = 0,\ u(1) = 1, \quad \varepsilon \ll 1.$$

(a) Find the inner and outer one-term expansion, then compute the composite expansion

(b) Derive a two-term composite expansion

(c) Find the exact solution of this problem.

(d) Plot one and two term composite expansion, as well as the exact solution on the same graph for $\varepsilon = 0.1$ and $a = -1$.

2. [Holmes, 2.2 #9] Consider the problem

$$4\varepsilon u_{xx} + 6\sqrt{x}u_x - 3u = -3, \quad u(0) = 0, \quad u(1) = 3, \quad \varepsilon \ll 1$$

(a) Find the inner and outer one-term expansion, then compute the composite expansion

(b) Compute a numerical solution using a boundary value problem solver and compare it with the asymptotic solution for two different well-chosen values of ε. Include a graph showing a visual comparison. Hint: you can use Maple's "dsolve" with option "numeric" for this. See an example of how to use it on the course website.

3. [Holmes, 2.3 #1] Consider the problem

$$\varepsilon u_{xx} + \varepsilon (x+1)^2 u_x - u = x - 1, \quad u(0) = 0, \quad u(1) = -1, \quad \varepsilon \ll 1.$$

Find a composite one-term expansion (there are two layers at two ends). Plot the composite expansion and compare with the numeric solution when $\varepsilon = 0.1,\ \varepsilon = 0.02$.

4. Consider the problem

$$\varepsilon u_{xx} - xu_x + u = 0,$$
$$u(-1) = 1, \quad u(b) = 2, \quad b > 1.$$

(a) This problem has boundary layers at both endpoints. Find a composite expansion for this problem. You will find that such expansion will depend on an arbitrary constant α that cannot be determined at this stage.

(b) Use an appropriate solvability condition to determine α as a function of b. Hint: first show (by direct verification) that the ODE

$$\varepsilon y'' + (xy)' + y = 0$$

admits a solution of the form

$$y = x \exp\left(-\frac{x^2}{2\varepsilon}\right).$$

(c) Compare the asymptotic solution obtained in part (b) with an exact numerical solution for two different well-chosen values of ε. Include a graph with visual comparison.

Remark: An alternative method to determine α is described in Holmes 2.2, probelm 3.