Integration

Consider $\quad f(z) = \int_z^\infty e^{-t}\, t^{-1}\, dt \quad , \quad z \to \infty$

Integrate by parts: $\quad f(z) = -e^{-t}\, t^{-1}\Big|_z^\infty - \int_z^\infty e^{-t}\, t^{-2}\, dt$

$$= e^{-z} z^{-1} - e^{-z} z^{-2} + \underbrace{2\int_z^\infty e^{-t}\, t^{-3}}_{R}$$

Now $\quad R = 2\int_z^\infty e^{-t}\, t^{-3} \le 2e^{-z}\int_z^\infty t^{-3} = e^{-z} z^{-2}$

$$= O(e^{-z} z^{-2})$$

So we get: $\quad f(z) \sim \dfrac{e^{-z}}{z} + O(e^{-z} z^{-2})$

[more precisely, $\quad \left| f(z) - \dfrac{e^{-z}}{z} \right| \le 2 e^{-z} z^{-2}$]

If we keep on integrating by parts further, we get

$$f(z) = e^{-z}\left(z^{-1} - z^{-2} + 2z^{-3} - 2\cdot 3 z^{-4} \;\ldots\; (N-1)!\, z^{-N} (-1)^{N+1} \right)$$

$$+ (-1)^N N! \int_z^\infty e^{-t}\, t^{-(N+1)}\, dt$$

$$= f_N + R_N$$

- Now for fixed z, note that $f_N \to \infty$ as $N \to \infty$ since $(N-1)!\, z^{-N} \to \infty$ as $N \to \infty$

- But for <u>fixed N</u>, $f_N \to f$ since $R_N \to 0$ as $\boxed{z \to \infty}$

- So f_N is a <u>divergent series</u>; but is still useful to estimate f by fixing N and letting $z \to \infty$.

- f_N ~~was~~ will give a good estimate initially but will eventually diverge as $N \to \infty$

<u>Ex</u>: fix $z = 5$, then we have:

$f(5) = 0.001148$ [using numerical integration]

$f_1(5) =$	0.001347
$f_2 =$	1078
3	1185
4	1121
5	1172
6	1121
7	1183
	$\vdots$

$f_5(5)$ is <u>closest</u> to $f(5)$

$\Rightarrow N = 5$ provides <u>best</u> estimate when $z = 5$.

In general, we should choose N in such a way that the last two terms of the series f_N are ~~nearly~~ equal in absolute value; ~~so that~~ such N corresponds to the critical point of $N \to f_N(z)$.

So we solve:

$$(N-1)!\, z^{-N} = (N-2)!\, z^{-(N-1)}$$

$$\Rightarrow \boxed{z = N-1} \Rightarrow$$

$$\boxed{N \sim z\,, \quad z \to \infty}$$

<u>Example 2</u>:

$$f(z) = \int_z^\infty \underbrace{e^{-t^2}}_{\frac{1}{-2t}\frac{d}{dt}(e^{-t^2})} dt = -\frac{1}{2t} e^{-t^2} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\, dt = \dots$$

$$\dots = \frac{e^{-z^2}}{2z}\left(1 - \frac{1}{2z^2} + \frac{1.3}{(2z^2)^2} - \frac{1.3.5}{(2z^2)^4} + \dots\right)$$

<u>Def</u>: We say that $\sum_{n=0}^{\infty} a_n \varphi_n(z)$ is an <u>asymptotic series</u> for $f(z)$ and we write:

$$f(z) \sim \sum_{n=0}^{\infty} a_n \varphi_n(z) \quad \text{as} \quad z \to z_0$$

if

$$\varphi_0 \gg \varphi_1 \gg \varphi_2 \gg \cdots \quad \text{as } z \to z_0$$

and

$$\sum_{n=0}^{N} a_n \varphi_n(z) \to f(z) \quad \text{as } z \to z_0$$

for all (fixed) N.

<u>Watson's Lemma</u>: Suppose that

- $f(t) = \sum_{0}^{\infty} a_n t^{\alpha_n}$ as <u>$t \to 0$</u>

  where $-1 < \alpha_0 < \alpha_1 < \cdots$

- $f(t) \leq C e^{\gamma t}$ for <u>all</u> t where

  C, γ are some positive constants.

Let $F(s) = \int_0^a e^{-st} f(t)\, dt.$

Then

$$F(s) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n+1)}{s^{\alpha_n+1}} \quad \text{as } \underline{s \to \infty}$$

Where $\Gamma(\alpha+1) \overset{\text{def}}{=} \int_0^{\infty} x^{\alpha} e^{-x}\, dx$ is the ~~Ga.~~ Gamma function.

<u>Remark</u>:

- Integration by parts yields:

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

and we have $\Gamma(1) = \int_0^{\infty} e^{-x} = 1$

so that $\boxed{\Gamma(n+1) = n!, \quad n \in \mathbb{N}.}$

- $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx = 2\int_0^{\infty} e^{-s^2} ds = \sqrt{\pi}$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Proof of Watson's lemma: Write

$$F(s) = \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{I_1} - \underbrace{\int_a^\infty e^{-st} f(t)\,dt}_{I_2}$$

Now for fixed N, we write:

$$I_1 = \sum_{n=0}^{N} a_n \int_0^\infty e^{-st} t^{\alpha_n} + \int_0^\infty t^{\alpha_{N+1}} e^{-st} p(t)$$

where $p(t) = O(1)$ as $t \to 0$

Note that $\displaystyle\int_0^\infty e^{-st} t^{\alpha}\,dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

and we estimate:

$$I_3 = \left|\int_0^\infty t^{\alpha_{N+1}} e^{-st} p(t)\right| \leq C\int_0^\varepsilon t^{\alpha_{N+1}} e^{-st} + \int_\varepsilon^\infty t^{\alpha_{N+1}} |p(t)| e^{-st}$$

$$\leq C \int_0^\infty t^{\alpha_{N+1}+1} e^{-st} + \int_\varepsilon^\infty |q(t)| e^{-st}$$

where $|q(t)| < C e^{\gamma t}$ for some γ;

So we get $\displaystyle\int_\varepsilon^\infty |q(t)| e^{-st} \leq C e^{-\varepsilon s}$

$$\Rightarrow I_3 \leq O\left(\frac{1}{s^{\alpha_{N+1}+1}}\right) + O\left(e^{-\varepsilon s}\right) = O\left(\frac{1}{s^{\alpha_{N+1}+1}}\right) \text{ as } s \to \infty$$

Similarly, $I_2 = O\left(e^{-as}\right) \ll O\left(\frac{1}{s^{\alpha_{N+1}+1}}\right)$

so that

$$F(s) = \sum_0^N \frac{\Gamma(\alpha_N + 1)}{s^{\alpha_N + 1}} + O\left(\frac{1}{s^{\alpha_{N+1}+1}}\right)$$

$$\Rightarrow F(s) \sim \sum_0^\infty \frac{\Gamma(\alpha_N + 1)}{s^{\alpha_N + 1}} \quad \text{as } s \to \infty$$

$\blacksquare$

Examples of Watson's Lemma

1) $$f(z) = \int_z^\infty e^{-t^2}\,dt \overset{u=t^2}{=} \int_{z^2}^\infty \frac{e^{-u}}{2\sqrt{u}}\,du$$

Let $u = z^2(1+t)$

$$\Rightarrow f(z) = \int_0^\infty \frac{ze^{-z^2}}{2} e^{-tz^2} (1+t)^{-\frac{1}{2}}\,dt$$

Now $$(1+t)^{-\frac{1}{2}} \underset{\substack{\uparrow \\ \text{Taylor}}}{=} \sum (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2^k k!} t^k, \quad |t|<1$$

So by Watson's lemma,

$$f(z) \sim \frac{ze^{-z^2}}{2} \sum_{k=0}^\infty \frac{1\cdot 3\cdot 5\cdots(2k+1)}{2^k\, z^{2k+2}} (-1)^k, \quad z\to\infty$$

We have re-obtained the result previously obtained using integration by parts (p. 3)

Laplace Transform

<u>Def</u>: $$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt .$$

<u>Thm</u>:

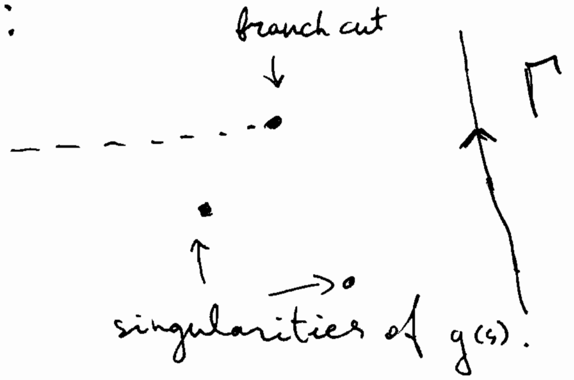
- $\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$
- $\mathcal{L}(f'') = s^2\mathcal{L}(f) - f'(0) - s f(0)$

Mellin Transform [inverse Laplace transform]:

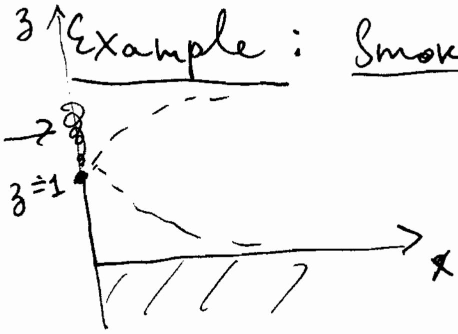
Let $g(s) = \mathcal{L}(f)(s)$. Then

$$f(t) = \frac{1}{2\pi i} \int_\Gamma e^{st} g(s)\, ds$$

where Γ is any vertical path in $\mathbb{C}$ to the right of any singularities / branch cuts of $g(s)$:



Example: Smokestack problem:



$$\begin{cases} C_x = C_{zz}, & z>0,\ x>0 \\ C_z = 0 & \text{on } z = 0 \end{cases}$$

$$C(0,z) = \delta(z-1)$$

- wind blowing to the right
- diffusion of pollutant in the z-direction
- $\delta(z-1)$ represents the pollutant coming out of a chimney at height 1 ;
- $z=0$ is the ground which reflects the pollutant $[c_z = 0$ at $z=0]$.

Let $g = \int_0^\infty e^{-sx} c(x,z)\,dx$; we obtain :

$$g_{zz} - s g = -\delta(z-1) \;;$$

$$\Rightarrow \quad \begin{cases} g_{zz} - s g = 0 , & z \neq 0 \\ g_z \big|_{1^-}^{1^+} = -1 , & g\big|_{z=1^-} = g\big|_{z=1^+} \\ g_z = 0 \text{ when } z = 0 ; & \\ g \text{ bdd} , & z \to \infty \end{cases}$$

$\Rightarrow$

$\Rightarrow$

$$g = \begin{cases} \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}} \left(e^{\sqrt{s}\, z} + e^{-\sqrt{s}\, z} \right), & 0 < z < 1 \\ \dfrac{e^{-\sqrt{s}\, z}}{2\sqrt{s}} \left(e^{\sqrt{s}} + e^{-\sqrt{s}} \right), & z > 1 \end{cases}$$

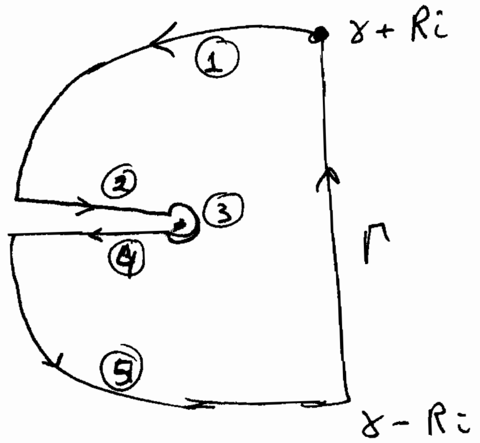
Now $\displaystyle c(x, z) = \int_{\gamma+i\infty}^{\gamma+i\infty} e^{sx} \, g(s, z)$;

In particular, when $z = 0$, we must find:

$$c(x, 0) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \, \frac{e^{-\sqrt{s}}}{2\sqrt{s}} \, ds, \qquad x > 0$$

Note that $\dfrac{e^{-\sqrt{s}}}{\sqrt{s}}$ has a <u>branch cut</u> at $s=0$.

So we deform Γ as shown:



$R \to \infty$

By Cauchy thm,

$$\int_\Gamma = -\int_{①+②+③+④+⑤}$$

Moreover $\displaystyle\int_{①,③,⑤} \to 0$ as $R \to \infty$

so we get $\displaystyle\int_\Gamma = -\int_{②} - \int_{③}$.

Along ②, we let $s = r e^{i\pi} \Rightarrow ds = -dr$

$$\sqrt{s} = r^{\frac{1}{2}} e^{\frac{i\pi}{2}} = i r^{\frac{1}{2}}$$

Along ④, we let $s = r e^{-i\pi} \Rightarrow ds = -dr$

$$\sqrt{s} = r^{\frac{1}{2}} e^{-\frac{i\pi}{2}} = -i r^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{2\pi i}\int_\Gamma e^{sx}\frac{e^{-\sqrt{s}}}{2\sqrt{s}}\,ds = \frac{1}{4\pi}\int e^{-rx}\frac{\left(e^{i r^{\frac{1}{2}}} + e^{-i r^{\frac{1}{2}}}\right)}{\sqrt{r}}\,dr$$

$$\Rightarrow \boxed{c(x,0) = \frac{1}{2\pi}\int_0^\infty e^{-rx}\frac{\cos\sqrt{r}}{\sqrt{r}}\,dr}$$

For large x we can expand $\frac{\cos\sqrt{\tau}}{\sqrt{\tau}} \sim \frac{1}{\sqrt{\tau}}$, $\tau \to 0$

so that applying Watson's lemma we obtain:

$$\boxed{C(x,0) \sim \frac{1}{2\pi}\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{x}} \sim \frac{1}{2\sqrt{\pi x}}}, \quad x \to \infty$$

Remark: For this pblm, we can find $c(x,0)$ explicitly even if $x = O(1)$: set $s = \sqrt{\tau} \Rightarrow$

$$c(x) = \frac{1}{\pi}\int_0^\infty e^{-s^2 x}\cos s \, ds$$

$$= \frac{1}{2\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} e^{-(s^2 x - is)}\, ds$$

Completing the square, we get

$$s^2 x - is = \left(s\sqrt{x} - \frac{i}{2\sqrt{x}}\right)^2 + \frac{1}{4x}$$

$$\Rightarrow \quad c(x,0) = \frac{1}{2\pi}\,\mathrm{Re}\left(e^{-\frac{1}{4x}}\int_{-\infty}^{\infty} e^{-\left(s\sqrt{x} - \frac{i}{2\sqrt{x}}\right)^2} ds\right)$$

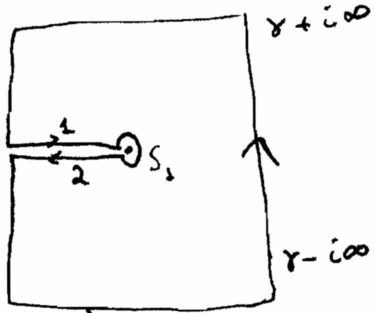
$$\boxed{c(x,0) = \frac{1}{2\sqrt{\pi x}}\, e^{-\frac{1}{4x}}}$$

Suppose that $F(s)$ has branch cut at $s = s_1$

so that ~~and that~~ $F(s) = a(s-s_1)^{\gamma} + o\left((s-s_1)^{\gamma}\right)$, $-1 < \gamma < 0$.

We ~~set~~ want to find $$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds, \quad \text{as } t \to \infty.$$

Deform the contour:



Along ①: $s = s_1 + r e^{i\pi}$

Along ②: $s = s_1 + r e^{-i\pi}$

$$\Rightarrow f(t) = \frac{-1}{2\pi i} e^{s_1 t} \int_0^{\infty} e^{-rt}\left(F(s_1 + r e^{i\pi}) - F(s_1 + r e^{-i\pi})\right) dr$$

Near $\underline{r = 0}$:

$$F(s_1 + r e^{i\pi}) = r^{\alpha} e^{i\pi\alpha} a + \cdots$$

$$- \quad F(s_1 + r e^{-i\pi}) = r^{\alpha} e^{-i\pi\alpha} a + \cdots$$

$$\overline{\qquad\qquad r^{\alpha} a \, 2i \sin(\pi\alpha)}$$

$$\Rightarrow f(t)$$

$$\sim -e^{s_1 t} \frac{a}{\pi} \sin \pi\alpha \int_0^{\infty} e^{-rt} r^{\alpha}\, dr$$

$$\boxed{f(t) \sim -e^{s_1 t} \frac{a}{\pi} \sin \pi\alpha \; t^{-(\alpha+1)} \, \Gamma(\alpha+1)}$$

Bessel ODE: Consider

$$\begin{cases} x y'' + y' + x y = 0 \\ y(0) = 1 \qquad y'(0) \text{ is bounded.} \end{cases}$$

We find: $\mathcal{L}(xy'') = \int e^{-sx} x y'' = -\frac{d}{ds}\left(\int e^{-xs} y''\right)$

$$= -\frac{d}{ds}\mathcal{L}(y'') = -\frac{d}{ds}\left(s^2 \mathcal{L}y - s y(0) - y'(0)\right)$$

Let $Y = \mathcal{L}(y)$; then

$$\mathcal{L}(xy'') = -s^2 Y_s - 2sY + 1$$

$$\mathcal{L}(y') = sY - 1$$

$$\mathcal{L}(xy) = -\frac{d}{ds}\mathcal{L}y = -Y_s$$

$$\Rightarrow \quad (1+s^2) Y_s + sY = 0 \Rightarrow (\ln Y)' = \left(-\tfrac{1}{2}\ln(1+s^2)\right)'$$

$$\Rightarrow \quad Y = c\,(1+s^2)^{-\frac{1}{2}} \quad \text{for some constant } c$$

To determine c, we expand $Y = \int_0^\infty e^{-sx} y(x)\,dx$ as $s \to \infty$ using Watson's lemma: $y(x) \sim 1$ as $x \to 0$

$$\Rightarrow Y \sim \frac{1}{s} \text{ as } s \to \infty$$

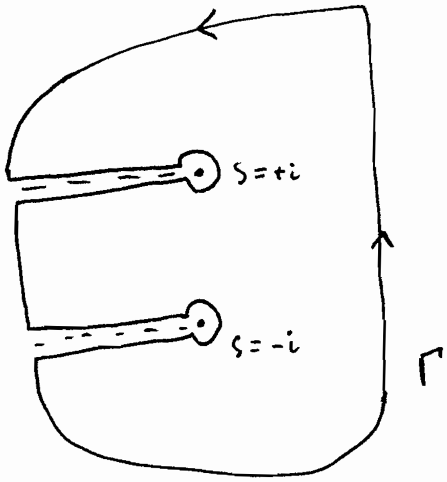
$$\Rightarrow \boxed{c = 1}$$

Thus we obtain:

$$y(x) = \frac{1}{2\pi i} \int_{\Gamma} e^{sx} (1+s^2)^{-\frac{1}{2}} \, ds$$

where Γ is to the right of $s = \pm i$

Note that $(1+s^2)^{-\frac{1}{2}} = (s+i)^{\frac{1}{2}} (s-i)^{\frac{1}{2}}$ has branch cuts at $s = \pm i$, so we deform Γ as shown:



Near $s = +i$ we get: $(s^2+1)^{-\frac{1}{2}} \sim (s-i)^{-\frac{1}{2}} \frac{1}{\sqrt{2}} e^{-\frac{i\pi}{4}}$, $s \sim i$

Near $s = -i$ we get: $(s^2+1)^{-\frac{1}{2}} \sim (s+i)^{-\frac{1}{2}} \frac{1}{\sqrt{2}} e^{\frac{i\pi}{4}}$, $s \sim -i$.

So applying the result from p. we obtain:

$$y(x) \sim - e^{ix} \frac{1}{\pi\sqrt{2}} e^{-\frac{i\pi}{4}} \sin\left(-\frac{\pi}{2}\right) x^{-\frac{1}{2}} \sqrt{\pi}$$
$$- e^{ix} \frac{1}{\pi\sqrt{2}} e^{\frac{i\pi}{4}} \sin\left(-\frac{\pi}{2}\right) x^{-\frac{1}{2}} \sqrt{\pi}$$

$$\boxed{y(x) \sim x^{-\frac{1}{2}} \cos\left(x - \frac{\pi}{4}\right) \sqrt{\frac{2}{\pi}}, \quad x \to \infty}$$

Note that $y(x)$ is precisely J_0 Bessel function.

To two orders, we obtain:

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left(z-\tfrac{1}{4}\pi\right) + \frac{1}{8z}\sin\left(z-\tfrac{1}{4}\pi\right)+\dots\right\}, \quad x\to\infty$$

To determine higher order corrections, we can also use an alternative method: We have $x^2y''+xy'+x^2y=0$

Substitute $y = x^{-\frac{1}{2}}e^{ix}z(x)$ to "peel off" the singularity; we obtain

$$z'' + 2iz' + \frac{1}{4}\frac{z}{x^2} = 0$$

as $x\to\infty$.

Next expand $z = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots$

Collecting terms in $\frac{1}{x}$ we get:

$\underline{O(x^{-n})}$: $\quad a_{n-2}(n-2)(n-1) - 2i(n-1)a_{n-1} + \frac{1}{4}a_{n-2} = 0$

$$\Rightarrow a_{n-1} = -i\,a_{n-2}\left(\frac{(n-2)(n-1)+\frac{1}{4}}{2(n-1)}\right), \quad n\geq 2$$

a_0 is arbitrary.

$$\Rightarrow a_0 = a_0, \quad a_1 = -\frac{i a_0}{8}, \quad a_2 = \frac{-9a_0}{128}, \quad a_3 = \frac{75}{1024}ia_0 \quad \dots$$

$$\Rightarrow y = A\frac{e^{ix}}{\sqrt{x}}\left(1-\frac{i}{8}\frac{1}{x} - \frac{9}{128}\frac{1}{x^2}\cdots\right)$$

or similarly,

$$y = B\frac{e^{-ix}}{\sqrt{x}}\left(1+\frac{i}{8}\frac{1}{x} - \frac{9}{128}\frac{1}{x^2}\cdots\right)$$

$$\Rightarrow y \sim \frac{\cos x}{\sqrt{x}}\left(1-\frac{9}{128}\frac{1}{x^2}\cdots\right)$$

or

$$y \sim \frac{\sin x}{\sqrt{x}}\left(\frac{1}{8}\frac{1}{x} - \frac{75}{1024}\frac{1}{x^3}+\dots\right)$$

Matching phase and first two constants, we get:

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \left\{ \cos\left(x - \frac{1}{4}\pi\right)\left(1 - \frac{9}{128} x^{-2} + O(x^{-4})\right) + \sin\left(x - \frac{1}{4}\pi\right)\left(\frac{1}{8} x^{-3} - \frac{75}{1024} x^{-5} + O(x^{-5})\right)\right\}$$

$x \to \infty$.

References:

- M. Ward, Course notes
- E.J. Hinch, Perturbation methods
- Bender & Orszag, Advanced mathematical methods for sicentists and engineers