Homework 6

1. An airplane must travel from point A to point B both at zero altitude and separated by distance d. Assume that the earth is flat. An airplane costs more money to fly at a lower altitude than at a higher one. The cost of traveling a distance ds at an altitude h is proportional to $e^{-h/H}ds$. Write down a first-order ODE for the airplane trajectory.

   Bonus: Using a numerical program of your choice, plot the resulting trajectory for some interesting parameter values of your choice.

2. Given a domain $D \subset \mathbb{R}^2$, define

$$I(f) = \int_D F(f, f_x, f_y)dxdy.$$

   where $f = f(x,y)$ and $f_x = \frac{\partial}{\partial x}f(x,y)$, $f_y = \frac{\partial}{\partial y}f(x,y)$. Show the the problem of minimizing $I(f)$ subject to fixed boundary conditions $f(x,y) = h(x,y)$ for $(x,y) \in \partial D$ yields the following Euler-Lagrange equation:

$$F_f - \frac{d}{dx}\left(F_{f_x}\right) - \frac{d}{dy}\left(F_{f_y}\right) = 0.$$

   Remark: you can use the "integration by parts formula"

$$\int\int_D \nabla u \cdot \vec{G}dxdy = \int_{\partial D} u\vec{G} \cdot \hat{n}dS - \int\int_D u\nabla \cdot \vec{G}dxdy$$

   where $\vec{G} = (G_1(x,y), G_2(x,y))$ is a vector function $\mathbb{R}^2 \to \mathbb{R}^2$, $u = u(x,y)$ is a scalar function $\mathbb{R}^2 \to \mathbb{R}, \partial D$ is the boundary of D, $\hat{n}$ is its normal, and dS is the surface element.

3.

   (a) Recall that the area of a surface given by the graph $z = f(x,y)$ above the region of the plane D is given by the double integral

$$I = \int\int_D \sqrt{1 + f_x^2 + f_y^2}dxdy$$

   Using question 2, show that minimizing I subject to fixed boundary constraint requires solving the PDE

$$f_{xx}\left(1 + f_y^2\right) + f_{yy}\left(1 + f_x^2\right) - 2f_xf_yf_{xy} = 0. \quad (1)$$

   (b) Show that the helicoid given by $z = \arg(x,y) = \arctan(y/x)$ satisfies (1).