Homework 7

1.

(a) Find the leading order behaviour of

$$\int_x^\infty e^{-t^3}\,dt, \quad x \to \infty \tag{1}$$

(b) Determine the two-term expansion for (1).

2. Consider the following problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0;$$
$$u(0,t) = 1; \quad u(x,0) = 0.$$

a) Use the Laplace's transform in t to solve this problem. Using Mellin (or inverse-Laplace) transfrom, express your answer in terms of an integral that does not involve any complex numbers.

b) From part (a), show that $u \to 1$ as $t \to \infty$ and with x fixed. Find the next-order expansion in the limit $t \to \infty$.

c) BONUS: Find the leading-order behaviour of $u(x,t)$ in the limit $t \to 0$.

3. Bessel K_n function has an integral representation given by

$$K_n(x) = \int_0^\infty e^{-x\cosh t}\cosh(nt)\,dt.$$

(a) Find the first term of the asymptotic expansion of K_n in the limit $x \to \infty$, for a fixed n.

(b) Find the first term of the asymptotic expansion of $K_n(pn)$ in the limit $n \to \infty$ for a fixed p.

4. Find the leading order expansion of

$$\int_0^1 \cos(xt^3)\tan(t)dt, \quad x \to \infty.$$

5. Bessel I_0 function is defined to be the solution to

$$x^2y'' + xy' - x^2y = 0, \quad y(0) = 1, \quad y'(0) \text{ is bounded.}$$

Use Laplace's transform method to determine the first term of the asymptotic behaviour of $y(x)$ the limit $x \to \infty$.

6. [BONUS] Find the first two terms of $\int_0^M \frac{\ln(x)}{1+x}dx$ in the limit $M \to \infty$. Hints: Try a substitution $x = e^t - 1$; to get the correction term, you will probably encounter $\int_0^\infty \ln(1-e^{-t})dt = \frac{-\pi^2}{6}$