Homework on Laplace Transorms

1. Find Laplace transforms of the following functions (Note: you should use Laplace transforms table, no integration required!):

$$
\begin{array}{ll}
(a) & f(t) = \sin(t)\exp(2t) \\
(b) & f(t) = \sin(t-a) \\
(c) & f(t) = \left\{ \begin{array}{rl} 0, & \text{if } t<1 \\ t-1, & \text{if } 1<t<2 \\ 1, & \text{if } t>1 \end{array} \right.
\end{array}
$$

2. (a) Suppose that $F(s) = \mathcal{L}\{f(t)\}$. Using the definition of Laplace's Transform, show that $F'(s) = -\mathcal{L}\{tf(t)\}$.

   (b) Use part (a) to easily find $\mathcal{L}\{t\sin t\}$ using the fact that $\mathcal{L}\{\sin t\} = 1/(s^2+1)$.

   (c) Find $F(s) = \mathcal{L}\{\frac{\sin t}{t}\}$. Hint: first compute $F'(s)$....

3. Find inverse Laplace transforms of the following functions.

$$
\begin{array}{ll}
(a) & F(s) = \dfrac{2}{3s+5} \\
(b) & F(s) = \dfrac{5s-6}{s^2-3s} \\
(c) & F(s) = \dfrac{e^{-3s}}{s(s^2+1)} \\
(d) & F(s) = \dfrac{s}{(s-3)\,(s^2+1)} \\
(e) & F(s) = \dfrac{s^3}{s^4+4} \quad (\text{Hint: } s^4+4 = \left(s^2-2s+2\right)\left(s^2+2s+2\right)\,) \\
(f) & F(s) = \ln\dfrac{s+a}{s-a} \quad (\text{Hint: take derivative...})
\end{array}
$$

4. Use Laplace transform to solve the following,

$$
\begin{array}{ll}
(a) & x'' - x' - 2x = 0, \quad x(0) = 0, \quad x'(0) = 2 \\
(b) & x' = x + 2y, \quad y' = x + e^{-t}, \quad x(0) = 0 = y(0)
\end{array}
$$

5. (a) Write the function $f(t) = \left\{ \begin{array}{rl} \pi, & t<\pi \\ 2\pi - t, & \pi < t < 2\pi \\ 0, & t > \pi. \end{array} \right.$ in terms of appropriate step functions.

   (b) Find $\mathcal{L}\{f(t)\}$.

   (c) Solve the initial value problem

$$x'' + x = f(t), \quad x(0) = 0, \quad x'(0) = 0.$$

   (d) You should verify that $x(\pi) = 2\pi,\ x(2\pi) = -\pi,\ \ x(3\pi) = \pi$. Sketch the solution.