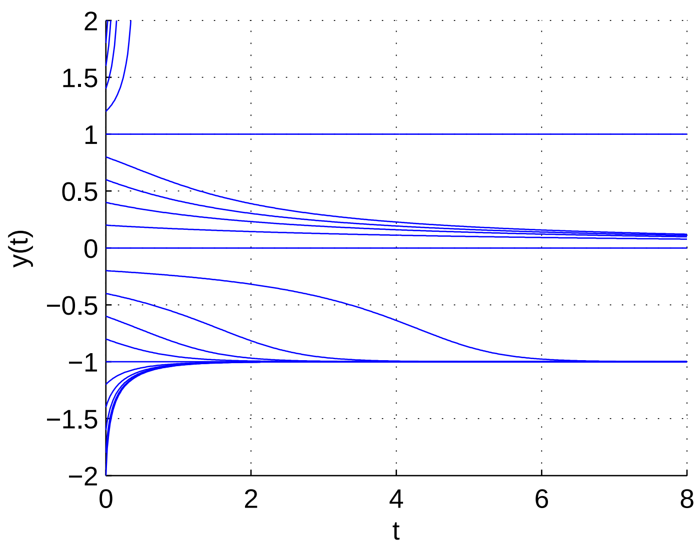
# MATH 2120, Homework 3

1. Consider the differential equation $y'(t) = y(y-1)(y-2)$. (a) Draw the phase diagram, find all steady states, and classify the critical points stable or unstable. (b) Find $\lim_{t\to\infty} y(t)$ for the solution with the initial condition $y(0) = 1.5$. (c) On the same graph, draw solutions corresponding to several well-chosen initial conditions that capture the variety of different possible behaviours.

2. The ODE $y'(t) = f(y)$ has the following solutions corresonding to various initial conditions from $y(0) = -2$ to $y(0) = +2$ in increments of 0.2:




   Give an example of $f(y)$ which can reproduce (qualitatively) this behaviour.

3. Consider the autonomous ODE
$$x' = x\,(ax - e^x)\,. \tag{1}$$

   (a) Note that there exists a special value $a_c$ such that the equation $ax = e^x$ has no solution if $0 < a < a_c$, and has two solutions when $a > a_c$ (with a double root when $a = a_c$). Compute the value of $a_c$. On the same graph, sketch $y = e^x$ and $y = ax$ for three values of $a$ : one where $0 < a < a_c$, one where $a = a_c$ and one where $a > a_c$.

   (b) Suppose that $0 < a < a_c$. In this case, sketch $x\,(ax - e^x)$ as a function of $x$ and show that the ODE (1) has only one equilibrium, $x = 0$. What is its stability? Draw the phase portrait and stability diagram for this case.

   (c) Suppose that $a > a_c$. In this case, sketch $x\,(ax - e^x)$ as a function of $x$ and show that the ODE (1) has three equilibria: one of them is $x = 0$ and there are two others, call them $x_2$ and $x_3$ with $x_2 < x_3$. Classify the stability of all three equilibria. Draw the phase portrait and stability diagram for this case.

4. Consider the ODE
$$y'(t) = y - t, \quad y(0) = 2. \tag{2}$$

   (a) Find the exact solution to this problem, and show that $y(1) \approx 4.71xx$. What are the next two digits $xx$?

   (b) Use Euler's method to approximate $y\,(1)$ with (1) $h = 0.5$ (2) $h = 0.25$; and (3) $h = 0.125$. For each $h$, record the value you obtained, as well as the error. Based on your data, comment on what happens to the the error each time you half $h$.

5. (a) Find the general solution to the ODE $y'' - y' - 2y = 0$. (b) Find the solution to this ODE subject to initial conditions $y(0) = 0$, $y'(0) = -1$.

6. (a) Find the general solution to the ODE $4y'' - 4y' + y = 0$. (b) Find the solution to this ODE that in addition satisfies initial conditions $y(0) = 1$, $y'(0) = 2$.

7. Find real numbers $x, y$ such that $z = x + iy$ where (a) $z = \frac{2}{3-4i}$. (b) $z = e^{3+i\pi/4}$.

8. (a) Let $z = 1 + \sqrt{3}i$. Find $r$ and $\theta$ such that $z = re^{i\theta}$. (b) What is $\left(1 + \sqrt{3}i\right)^{2015}$?

9. (a) Find the general solution to the ODE $y'' - 4y' + 5y = 0$. (b) Find the solution to this ODE that in addition satisfies initial conditions $y(0) = 1$, $y'(0) = 2$.

10. Find the general solution to the ODE $y'' + \frac{y'}{x} - \frac{4}{x^2}y = 0$. Hint: first try the solution of the form $y = x^p$ for some $p$.

11. A mass of $m = 0.1$ kg is attached to a spring. The spring constant is known to be $k = 50N/m$, and you wish to find the friction constant $c$; recall that the mass motion satisfies $mx'' + cx' + kx = 0$. To find the spring constant, you observe the spring oscillates with a frequency of 3 Hertz. Determine $c$.