MATH 2120, Homework 5

1. Find the general solution to the system $x' = Ax$, where A is as specified below. Make sure to write the solution in purely real form.

   (a) $A = \begin{pmatrix} 1 & 0 \\ -4 & 3 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$.

   (c) $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$

   (d) $A = \begin{pmatrix} -5 & -3 \\ 3 & 1 \end{pmatrix}$

2. A 3x3 real matrix has three eigenvalues. One of them is $\lambda_1 = -1$ and its corresponding eigenvector is $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Another is $\lambda_2 = 1 + i$ and its corresponding eigenvector is $v_2 = \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix}$. (a) What is the third eigenvalue and its corresponding eigenvector? (b) Find the general solution to $x' = Ax$. (c) Find the solution to $x' = Ax$ subject to initial condition $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

3. Two tanks are connected by two pumps: one pump pushes the liquid from tank A to tank B at the rate of 6 liters/hour while the other pushes from tank B to tank A at the same rate. Initially, both tanks contain 2 liters of liquid and tank A contains pure water while tank B has a mixture of 80% water and 20% pollutant. (a) Find the concentration of pollutant in tanks A and B after ten minutes. (b) Find the concentration of pollutant in tanks A and B after a very long time.

4. Classify stability (i.e sync/source, spiral, saddle, center...) and sketch the phase portait of the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where

   (a) $A = \begin{pmatrix} 1 & 0 \\ -4 & 3 \end{pmatrix}$

   (b) $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

   (e) $A = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}$

5. Consider system $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where $A = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$. Classify stability and sketch the phase portait as a is increased from -3 to $+3$ for several well-chosen values of a. What happens when a crosses through zero? What happens when a crosses through ± 2?