MATH 2120, Homework 7

1. A two-mass, three-spring system consists of two masses at positions $(x(t), y(t))$ attached by a spring to each other and to two adjacent walls. An example of such a system (with a particular choice of masses and spring constants to make sure the solution will not have any square roots) is

$$x'' = -2x + \frac{3}{2}y$$
$$y'' = \frac{4}{3}x - 3y.$$

(a) Rewrite this system as a four-dimensional first-order linear system of the form $\vec{x}' = A\vec{x}$ where $x = (x_1, x_2, x_3, x_4)^t$ with $x_1 = x, \quad x_2 = y, x_3 = x', \ x_4 = y'$.

(b) Find the general solution for this system (in terms of real functions).

2. A 3x3 real matrix A admits an eigenvalue $\lambda = 2 + i$ with the corresponding eigenvector $[1 + i, 0, 1]$ and an eigenvalue $\lambda = -1$ with the corresponding eigenvector $[1, 1, 0]$. Find $\exp(At)$.

3. Consider the system

$$x' = y, \quad y' = ay + x^2 - x.$$

(a) Determine the equilibrium locations.

(b) Consider the case $a = 0$. Linearize around the equilibria and determine their stability properties. Then sketch the phase portrait for this case.

(c) Show that the origin is stable if $a < 0$ and is unstable if $a > 0$. Sketch the phase portrait for $a = 0.1$ and for $a = -0.1$.

(d) What happens when $a = 2$? Sketch the phase portraits for $a = 1.9, \ a = 2, \ a = 2.1$.

4. The following predator-prey system

$$x' = x(2 - x - y)$$
$$y' = y(-1 + x)$$

models the interaction the populations of rabbits and wolves, represented by $x(t)$ and $y(t)$, respectively.

   (a) Sketch nullclines and determine equilibrium points.
   (b) For each equilibrium point, determine its local stability. Sketch the phase portrait of this system.
   (c) What conclusions can you draw from your phase plane analysis?

5. Schnakenberg (1979) considered the following simplified model of glycolysis:

$$\frac{dx}{dt} = x^2y - x, \qquad \frac{dy}{dt} = a - x^2y$$

where $a > 0$. Here, x and y are chemical concentrations and are assumed to be positive.

(a) What is the steady state of the system? Compute the linearization (Jacobian) at the steady state.

(b) As the parameter a varies, the steady state of the system changes its behaviour. Does the system undergo a Hopf bifurcation for any value(s) of a? If so, at what value(s) does the Hopf bifurcation(s) occur?

(c) Describe the dynamics of the system for various (well-representative) values of a. You may use a computer to help plot the phase portrait. Google pplane for instance.