Practice questions for midterm

1. Review all the homework questions (up to and including hw4).

2. Solve the following first-order ODE's.

$$(a)\ \ y' = \frac{y}{2x}$$
$$(b)\ y' - 2y = 3e^{-3x}, \quad y(0) = 3$$
$$(c)\ 2x\ y' - y = 2x^{1/2}, \quad y(1) = 2$$
$$(d)\ 2xyy' + 2x + y^2 = 0.$$
$$(e)\ 2yy' + 2x + y^2 = 0.$$
$$(f)\ y' = (x - y)^2$$

3. A homicide victim was found in a room that is kept at a constant temperature of 20°C. The body temperature was immediately measured and found to be 26 degrees. One hour later, the temperature was again recorded and found to be 24 degrees. Assuming that the victim's temperature was 37 degrees just before death, determine how long the body was dead before it was discovered.

4. Consider a reservoir with a volume of 8 billion cubic meters and and initial pollutant concentration of 0.25%. There is a daily inflow of 500 million cubic meters of water with a pollutant concentration of 0.03% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.1%?

5. For an initial problem $y' = xy - y,\ y(2) = 2$, use Euler's method to estimate $y(4)$ using two time-steps.

6. Consider the differential equation $y'(t) = y \sin y$. (a) Draw the phase diagram, find all steady states, and classify their stability. (b) On the same graph, draw solutions corresponding to several well-chosen initial conditions that capture the variety of different possible behaviours. Find $\lim_{t\to\infty} y(t)$ for solutions corresponding to initial values $y(0) = 1, \quad y(0) = -1$ and $y(0) = 7$.

7. (a) For what value of k does the ODE $2x^2 + ky + (x^2y - x)y' = 0$ have an integrating factor of the form $r(x, y) = r(x)$? (b) For the value of k that you found in part (a), find $r(x)$ and the solution to the ODE subject to initial condition $y(1) = 2$. You may leave your answer in implicit form.

8. Suppose that the differential equation $y' = f(x, y)$ satisfies the existence and uniqueness theorem for all values of x and y. Assume that $y_1(x) = x$ and $y_2(x) = x + 2e^{-x^2}$ are two solutions to the differential equation $y' = f(x, y)$. Let $g(x)$ be the solution to the initial value problem:

$$\begin{cases} y' = f(x, y), \\ y(0) = 1. \end{cases} \tag{1}$$

Find the limit $\displaystyle\lim_{x\to\infty} \frac{g(x)}{x}$.

9. Find the solution to the following problems (or general solution if no initial conditions are given)

$$(a)\ \ y'' - 4y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 0$$
$$(b)\ \ y'' + y' = \cos(t), \quad y(0) = 0, \quad y'(0) = 0$$
$$(c)\ \ y'' + y' = e^t + e^{-t}$$
$$(d)\ \ y'' - 2y' + y = xe^x$$

10. A periodically forced spring is modelled as

$$y'' + 2y' + 2y = 5\cos t. \tag{2}$$

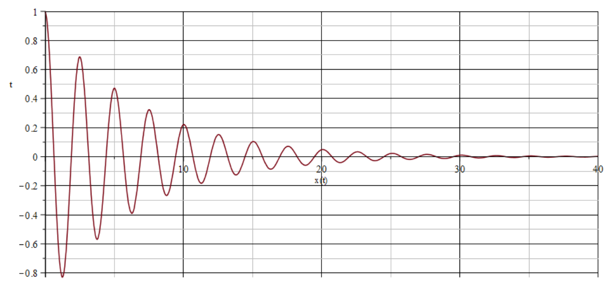
(a) Find the general solution of (2).

(b) Initially, the spring is at rest, that is $y(0) = 0 = y'(0)$. Find the spring's position at a later time t.

11. (a) The spring displacement $x(t)$ of unforced spring is modelled as

$$my'' + cy' + ky = 0. \tag{3}$$

The mass m is known to be 2kg. The string displacement is measured and is found to be as shown below:

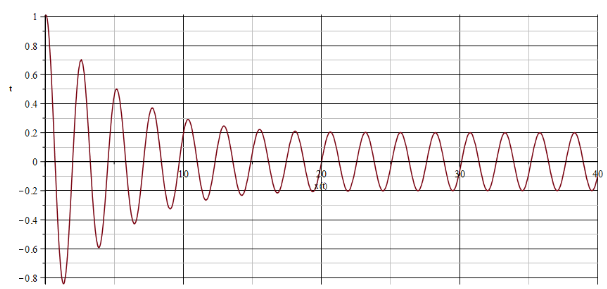


Determine c and k.

(b) The spring is being periodically forced according to the model

$$y'' + cy' + ky = F\sin(\omega t). \tag{4}$$

Here, $m = 2$ and c and k are as you found in part (a). The string displacement is measured and is found to be as shown below:



Determine F and ω.