Review/practice questions final exam

Topics covered:

- First-order ODE's:
  - Solving: separable, linear (nonhomogeneous), exact, integrating factors
  - Mixing problems, Newton's law of cooling
- Linear ODE's:
  - Characteristic equation, general solution, initial conditions
  - Nonhomogeneous linear ODE: finding particular solution using the method of undetermined coefficients
  - Forced spring, resonance
- Linear systems of ODEs
  - Finding eigenvalues/eigenvectors
  - Degenerate case: generalized eigenvector
  - Matrix exponential
  - Nonhomogeneous systems
  - Sketching phase portraits of linear systems,
- Qualitative theory of nonlinear systems
  - Steady states, linearization, stability
  - Sketching 2d systems: nullclines, stability of equilibria, phase portraits, bifurcation theory
- Laplace transforms
  - Using table to find Laplace transform or inverse Laplace transform
  - Partial fraction decomposition (various cases)
  - Piecewise-defined functions: rewrite in terms of step functions and Laplace...
  - Solving linear ODE's using Laplace transforms

Advice on how to study for exam:

- Do the sample questions on this handout; go over the homework questions and sample midterm and midterm.
- Go over the solutions to homeworks, midterm, and practice questions (posted on the course website, www.mathstat.dal.ca/~tkolokol/classes/ode1)

- Concentrate on the material that you have had trouble with, as diagnosed by homeworks/midterm.

Some additional questions (NOTE: this is in ADDITION to questions in hw/midterm)

1. A tank is filled with 100 liters of pure water. Salt solution at a concentration of 0.1kg/liter enters the tank from the top at 4 liters per minute, and exits from the bottom at 1 liter per minute. How much salt is in the tank when it has 400 liters of water?

2. Find the general solutions to the following ODE's:
$$(i)\ y' = \frac{y}{x} + xy^2 \qquad (ii)\ xy' + y = 2e^{2x}$$
$$(iii)\ \ 2xy^3 + e^x + \left(3x^2y^2 + \sin y\right) y' = 0$$

3. Find the integrating factor and solve the ODE
$$3x^3y + x^2y^2 + \left(x^4 + x^3y\right) y' = 0$$

4. Solve $y'' + 4y' + 13y = 0, \quad y(0) = 2,\ y'(0) = 5.$

5. Find a third order linear ODE whose solutions are e^{-x}, e^x and xe^x.

6. Find a particular solution to
$$y'' + 3y' = e^{-3t}.$$

7. Two tanks are connected by two pumps: one pump pushes the liquid from tank A to tank B at the rate of 2 liters/minute while the other pushes from tank B to tank A at the same rate. Initially, both tanks contain 5 liters of liquid and tank A contains pure water while tank B has a mixture of 80% water and 20% pollutant.

   (a) Find the concentration of pollutant in tanks A and B after one minute.

   (b) Find the concentration of pollutant in tanks A and B after a very long time.

8. Let $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$.

   (a) Compute e^{tA}.

   (b) Write down e^{-tA}.

   (c) Find the solution to the system $x' = Ax + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$ with $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

9. A 2x2 matrix A has eigenvalues $\lambda = 1, -2$ and the corresponding eigenvectors $[1, 0],\ [1, 1]$. Sketch the phase portrait of the system $x' = Ax$.

10. Find e^{tA} where (i) $A = \begin{bmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ 3 & 1 & 2 \end{bmatrix}$. and (ii) $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

11. A 3x3 real matrix A admits an eigenvalue $\lambda = i$ with the corresponding eigenvector $[1+i, 0, 1]$ and an eigenvalue $\lambda = 2$ with the corresponding eigenvector $[0, 1, 0]$. Find $\exp(tA)$.

12. Find the solution to $x' = Ax + f(t),\ x(0) = x_0$ where A is as in (c), $f(t) = [0, e^{2t}, 0]$ and $x_0 = [0, 0, 1]$.

13. (a) Rewrite the ODE
$$y'' + y = f(t)$$
as a system of first order ODE's. Then use the fundamental solution technique to compute $y(t)$ subject to initial conditions $y(0) = 0 = y'(0)$. Conclude that
$$y(t) = \int_0^t \sin(t-s) g(s).$$
(b) What is the solution to $y''+y = f(t)$ subject to arbitrary initial conditions $y(0) = y_0$ and $y'(0) = v_0$?

14. You are given a real $2x2$ matrix A whose eigenvalues are $\lambda = 2 \pm i$. Moreover it is known that the eigenvector corresponding to $\lambda = 2 + i$ is given by $v = \begin{pmatrix} 1 \\ i+1 \end{pmatrix}$.
    (a) Determine the general solution to the system $x' = Ax$.
    (b) Sketch the phase portrait of the system $x' = Ax$.

15. A model of fish population under a constant harvest pressure is:
$$\frac{dy}{dt} = y(1-y) - h$$
where $y(t)$ represents the population of fish and $h \geq 0$ is the rate of harvesting. Analyse this model for different harvest rates h. What are the equilibria? What is their stability? Make sure to include phase diagram for various values of h. Describe any changes in the stability diagram that occurs as h is increased from zero. What are the implications for fisheries?

16. (a) Determine all the critical points (steady states) of the system
$$x' = 2 - x^2 - y^2$$
$$y' = x^2 - y^2$$
(b) Classify the linear stability of each critical point
(c) Sketch the phase portrait.

17. Consider the system
$$x' = y + ax - xy^2$$
$$y' = ay - x - y^3.$$
Show that the origin undergoes a Hopf bifurcation as a is increased past zero. Sketch the phase portrait for $a = -0.1$, $a = 0$ and $a = 0.1$.

18. Find the Laplace transform of the function

$$f(t) = \begin{cases} 3 - t, & t \leq 3 \\ 0, & t \geq 3. \end{cases}$$

19. Solve the ODE $x'' + 2x' + 2x = f(t)$, subject to initial conditions $x(0) = 0 = x'(0)$ where $f(t) = \begin{cases} 1, & t \leq 2 \\ 0, & t \geq 2 \end{cases}$. Sketch the graph of the solution.