1. Solve the PDE system

$$\begin{aligned} u_t &= 3u_{xx} \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 5\sin(x) + 2\sin(5x). \end{aligned}$$

2. (a) Solve the following heat equation:

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(0,t) &= 0; \\ u(L,t) &= 0; \\ u(x,0) &= f(x) \quad \text{for } 0 < x < L. \end{aligned}$$

Express your answer in terms of an appropriate Fourier series and write down coefficients in terms of certain integrals of $f(x)$.

(b) Suppose $f(x) = 1$. Find explicitly the Fourier coefficients.

(c) With $f(x) = 1$ and $L = 1$, sketch $u(x,t)$ when $t = 1$. Hint: how many terms of the series would you need for an accurate sketch? What is the value of $u(0,1)$?

(d) With $f(x) = 1$ and $L = 1$, estimate (using calculator/computer) the value of t such that $u(0,t) = 1/2$. How many terms did you need to accurately estimate this?

3. Use separation of variables to find the solution to

$$\begin{aligned} u_t &= u_{xx} - u \\ u_x &= 0 \text{ at } x = 0, \quad u = 0 \text{ at } x = 1 \\ u &= f(x) \text{ at } t = 0. \end{aligned}$$

You should express your solution in terms of certain integrals of $f(x)$.

4. (4.6.10) Suppose that the wire is circular and insulated, so there are no ends. You can think of this as simply connecting the two ends and making sure the solution matches up at the ends. That is, find a series solution of

$$\begin{aligned} u_t &= ku_{xx}, \\ u(0,t) &= u(L,t); \\ u_x(0,t) &= u_x(L,t); \\ u(x,0) &= f(x) \quad \text{for } 0 < x < L. \end{aligned}$$

Express your answer in terms of an appropriate Fourier series and write down coefficients in terms of certain integrals of $f(x)$.

5. (a) Solve the problem

$$\begin{aligned} u_t &= 3u_{xx} + \sin(4x) \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 0. \end{aligned}$$

(b) By reusing your answers to questions 5(a) and 1, write down the solution to the question

$$\begin{aligned} u_t &= 3u_{xx} + \sin(4x) \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 5\sin(x) + 2\sin(5x). \end{aligned}$$

6. (a) Express the function $f(x) = 1$ in terms of a sine Fourier series on an interval $(0, 1)$. In other words, write

$$1 = \sum_{n=1}^{\infty} c_n \sin(\pi n x), \qquad x \in (0, 1)$$

and then determine explicitly c_n.

(b) Solve the problem

$$\begin{aligned} &u_t = u_{xx} + 1 \\ &u(0, t) = 0, \quad u(1, t) = 0, \\ &u(x, 0) = 0. \end{aligned}$$

HINT: Use part (a).

(c) Let $u_\infty(x) = \lim_{t\to\infty} u(x, t)$. Show that $u_\infty(x) = -\frac{x^2}{2} + \frac{x}{2}$.