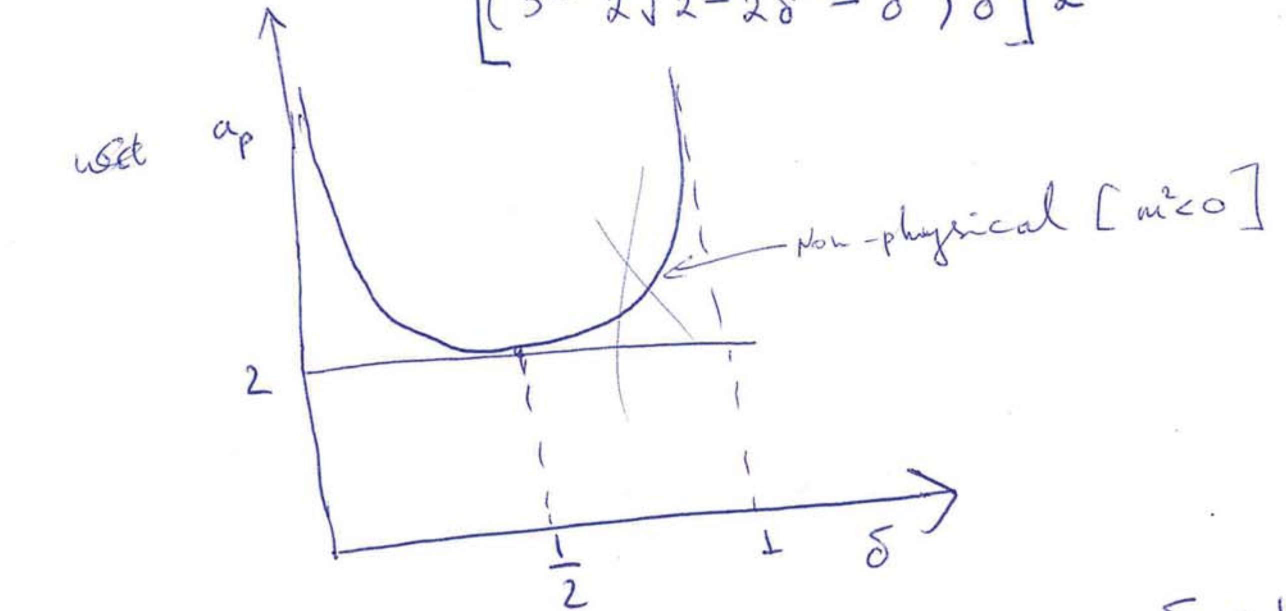
In general, the Turing instability first appears as a is decreased below $a = a_p$; the value of a_p is determined using a <u>double-root condition</u>: $\begin{cases} \lambda = 0 \\ \dfrac{\partial \lambda}{\partial m} = 0 \end{cases}$

which yields, after some algebra,

$$a_p = \frac{3 - 2\sqrt{2 - 2\delta}}{\left[\left(3 - 2\sqrt{2-2\delta} - \delta\right)\delta\right]^{\frac{1}{2}}} \quad , \quad 0 < \delta < \frac{1}{2}$$

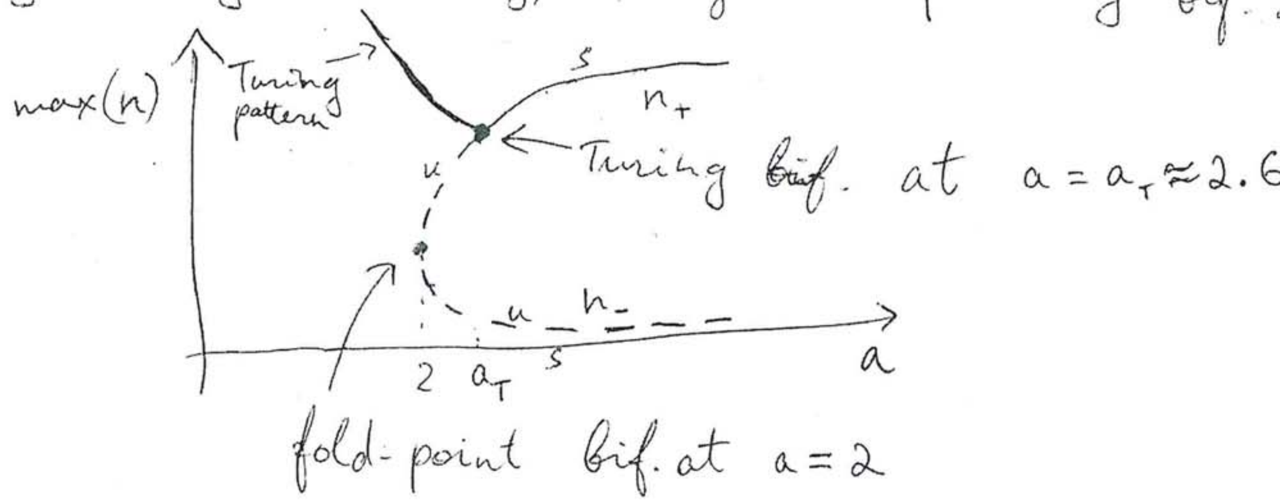


$\Rightarrow$ Turing instability exists as long as $0 < \delta < \frac{1}{2}$

[no Turing instability if $\delta > \frac{1}{2}$].

- Incorporating Turing instability, we get the following bif. diagram

$\underline{\delta = 0.04}$:



- Turing bif. leads to spatially inhomogeneous patterns
- These patterns extend well beyond the fold-point and allow vegetation to survive in "patches" with limited precipitation
- See FlexPDE script for numerical simulations of this.

Turing instability, slow drift and noise [1]:

- Imagine there is a gradual decrease in precipitation so that

$$a = a_0 - \varepsilon t + \text{"noise"} \quad \left[\text{or } a = a_0 + \frac{a_1 - a_0}{2}\left(1 - \cos(\varepsilon t)\right) + \text{noise}\right]$$

where $a_0 > a_p$ is <u>above</u> Turing bifurcation point

- If i.c. are homogeneous and there is no noise, then the Turing bifurcation never gets activated!
- Even when the noise is present, it takes some time for the Turing patterns to grow; there is a <u>delay in the bifurcation</u>.
- Depending on the size of the delay, if the delay is too large, you may fall off the fold point $a_c = 2$ <u>before</u> the system has a chance to transition to patterned state! [causing extinction]
- Mathematical analysis (see below) shows that extinction is <u>more likely</u> if precipitation decreases more rapidly [i.e. bigger ε, or ...]

[1] Y. Chen, T. Kolokolnikov, J. Tzou, and C. Gai, Patterned vegetation, tipping points, and the rate of climate change, EJAM 2015

Analysis:

$$\begin{cases} \frac{dn}{dt} = \delta\, n_{xx} - n + n^2\omega \\ 0 = \omega_{xx} - \omega + n^2\omega + a(\varepsilon t) + \sigma_0 \frac{dW}{dt} \end{cases}$$

where $a(s) =$ is a decreasing function with
$a(0) > a_p$ and $a(\varepsilon\, t_{end}) < a_p$

eg. $a(s) = a_0 - s$, with $a_0 > a_p$

and $dW = \sum_m \sqrt{dt}\, \xi_m(t) \exp(imx)$, $\xi_{-m} = \overline{\xi}_m$

$\xi_m(t) \equiv$ Gaussian random variable with mean 0, std. 1

- This is a spatio-temporal Weiner process
- See [1] for details and matlab code.

Recall that the quazi-steady state is given by

$$n_\pm = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}, \quad \omega_\pm = \frac{1}{n_\pm} \quad ; \quad a = a(\varepsilon t), \; n_\pm = n_\pm(\varepsilon t), \; \omega_\pm = \omega_\pm(\varepsilon t)$$

We linearize around n_+, ω_+:

$$n(x,t) = n_+(\varepsilon t) + \varphi(x,t), \quad \omega(x,t) = \omega_+(\varepsilon t) + \psi(x,t), \text{ with } \varphi, \psi \ll 1$$

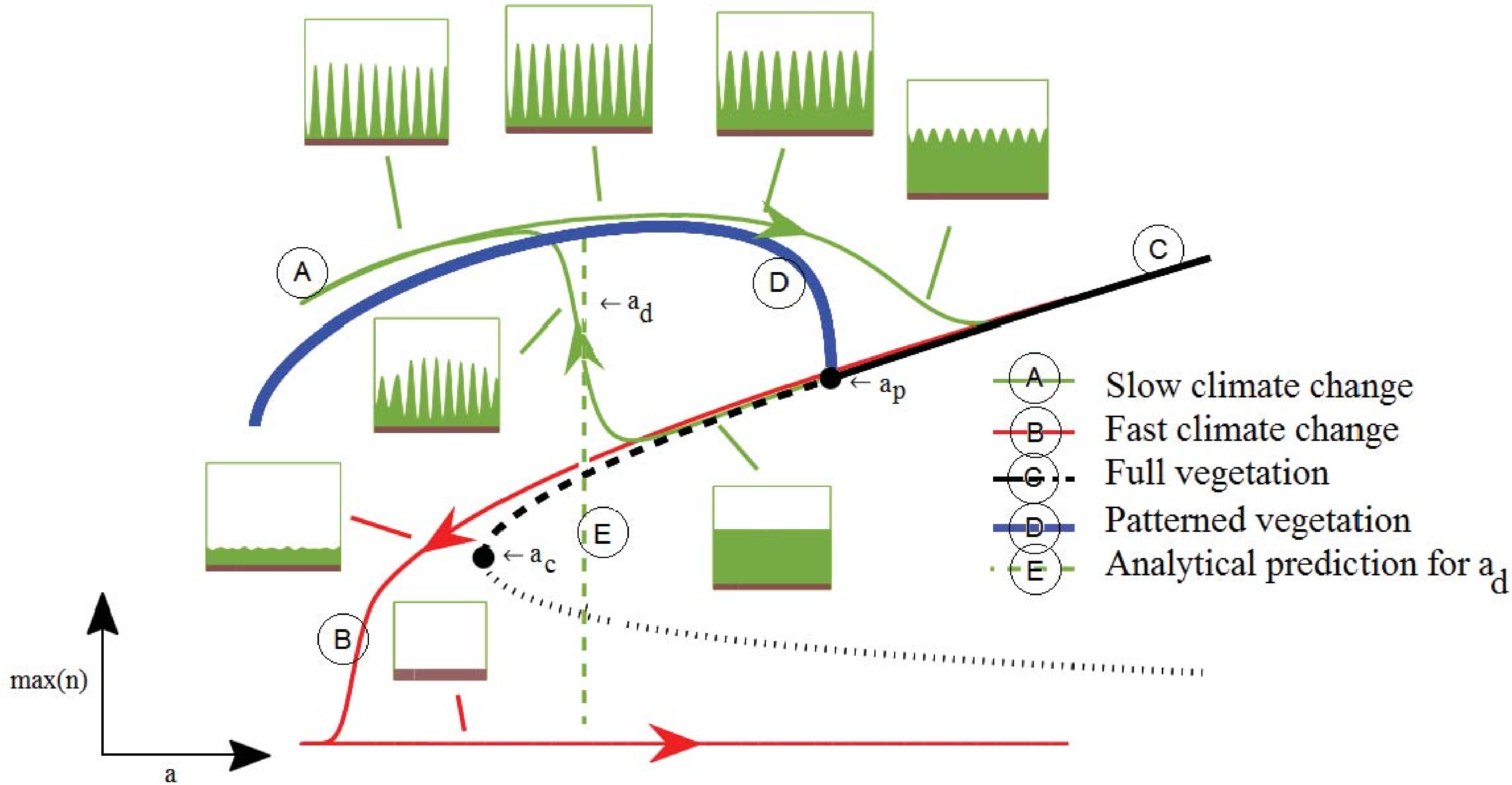


FIG. 1. (colour online) Evolution of vegetation patterns in the presence of slow precipitation drift. The green curve A avoids the tipping point by going into a reversible patterned state. The red curve B falls off the fold point leading to irreversible extinction. Model (1) is used with a given by (5). The Green curve A corresponds to slow decrease ($\varepsilon = 0.006$, corresponding to a change of 2 mm/year) while the red curve B corresponds to a faster decrease ($\varepsilon = 0.03$, corresponding to a change of 40 mm/year). Other parameters are $a_0 = 2.85, a_1 = 1.75, \sigma_0 = 0.0001, \delta = 0.05, d = 1, c = 0$ on a domain of size $L = 22.839$ with periodic boundary conditions. The full vegetation state (black curve C) bifurcates into a patterned state (blue curve D, computed by the software package AUTO [12]) at $a = a_p = 2.4743$, and has a fold point at $a_c = 2$. For values of a below a_p and above a_c, the full vegetation state still exists (dashed black curve C) but is unstable to spatial perturbations. The vertical green dashed line E at $a = a_d = 2.18$ is the asymptotic prediction for the delayed "take-off" value of the green curve. Excellent agreement between the asymptotics and numerics is observed.

$$\begin{cases} \varepsilon n_+'(\varepsilon t) + \varphi_t = \delta \varphi_{xx} - \varphi + 2\varphi n_+ \omega_+ + n^2 \psi \\ 0 = \psi_{xx} - \psi - 2\varphi n_+ \omega_+ - n_+^2 \psi + \sigma_0 \dfrac{dW}{dt} \end{cases}$$

Decompose $\begin{cases} \psi = \sum \psi_m(t) e^{imx} \\ \varphi \quad\ \ \varphi_m \end{cases}$, $dW = \sum \sqrt{dt}\, \xi_m\, e^{imx}$

Dropping the "m" we get; with $m \neq 0$:

$$\begin{cases} \varphi_t = -\delta m^2 \varphi - \varphi + 2\varphi n_+ + n_+^2 \psi \\ 0 = -m^2 \psi - \psi - 2\varphi n \omega - n^2 \psi + \dfrac{\sigma_0 \sqrt{dt}\ \xi}{dt} \end{cases}$$

$$\searrow \psi = \frac{2\varphi n \omega - \sigma_0 \dfrac{\sqrt{dt}}{dt} \xi}{m^2 + 1 + n^2}$$

$$\searrow \boxed{\frac{d\varphi}{dt} = \alpha \varphi + \beta \frac{\sqrt{dt}}{dt} \xi}$$

Where
$$\begin{cases} \alpha(\varepsilon t) = \underbrace{-\delta m^2 - 1 + 2n_+} + \dfrac{2 n_+ \omega_+ \, n_+^2}{m^2 + 1 + n_+^2} \\ \beta(\varepsilon t) = \dfrac{\sigma_0 n_+^2}{m^2 + 1 + n_+^2} \end{cases}$$

$\ldots dt\ (1) + \beta(\varepsilon t) \sqrt{dt}\ \xi\ ]$

Introduction to SODE

Simplest SODE:

- $dx = 1\sqrt{dt}\,\xi$, here $\xi \equiv$ Gaussian ;
- Matlab implementation :

```
dt = 0.01;
x = 0
for t=0:dt:5
    x = x + sqrt(dt)*randn
end
```

- Sol'n is interpreted in terms of probability density:

- Let $u(x,t) \equiv$ prob. density; that is:

$$\int_{x_1}^{x_2} u(x,t)\,dx \equiv \text{probability that } x_1 \le x(t) \le x_2 .$$

- Key property of $u(x,t)$:

$$u(x, t+\Delta t) = \int_{-\infty}^{\infty} \underbrace{G(x,y)}_{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}}} u(x+\sqrt{dt}\,y, t)\,dy$$

- Expand in Taylor series:

add

$$u(x,t) + \Delta t\, u_t + O((\Delta t)^2) = \int G(x,y)\left(u + u_x\sqrt{dt}\,y + \frac{u_{xx}\,dt\,y^2}{2} + O((\Delta t)^{3/2})\right)dy$$

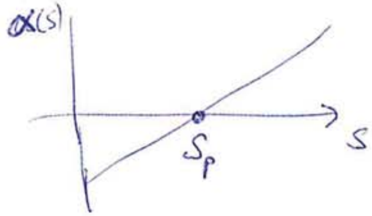
$$= u + u_{xx}\,dt + \cdots$$

$\Rightarrow$ Let $\Delta t \to 0$:

$$\boxed{u_t = \frac{u_{xx}}{2}} \longleftarrow \text{"Fokker-Plank Eq'n"}$$

Now consider the SDE $\begin{cases} \dfrac{dx}{dt} = \alpha(\varepsilon t)\, x + \dfrac{\sqrt{dt}\ \xi}{dt}\, \beta(\varepsilon t), \\ x(0) = 0 \end{cases}$, $\boxed{\varepsilon, \sigma_0 \ll 1}$

Suppose $\alpha(s)$ is something like



i.e. $\begin{cases} \alpha(0) < 0; & \alpha(s) \text{ is increasing}, \\ & \alpha(S_p) = 0 \\ & \alpha'(S_p) < 0 \end{cases}$ eg. $a(s) = -1 + s$, $S_p = 1$

- In the absence of noise ($\beta = 0$), the ODE $x' = \alpha(\varepsilon t)\, x$ has a s.s. $x = 0$ which is "stable" if $\alpha < 0$, "unstable" if $\alpha > 0$.

- If $\beta \ll 1$ and a is increased slowly, it takes some time before instability is "fully realized"
i.e. $x(t)$ may remain small even for $\varepsilon t > S_p$

Back to $\boxed{d\varphi = \alpha\varphi\, dt + \beta\sqrt{dt}\,\xi \,,\ \varphi(0) = 0}$

- The Turing bifurcation corresponds to $\alpha = 0 = \frac{\partial\alpha}{\partial m}$;
- with $\alpha < 0$ when s.s. is stable.

<u>Ex</u>: Suppose $\alpha(\varepsilon t) = -1 + \varepsilon t$; take $\beta(\varepsilon t) = 10^{-5}$, $\varepsilon = 0.02$
Then $\varphi = 0$ is "stable" when $\alpha < 0$, "unstable" if $\alpha > 0$
<u>However</u> the instability is only "observed" when $\alpha \approx 0.65$,
<u>much later</u>, than $\alpha = 0$ - see Matlab example.

- We define "observed" by the time at which $|\varphi| > 1$ [for numerical purposes].

Q: - <u>Better definition</u>: the "take-off" or "blowup" time is when the <u>variance of the density becomes "very big".</u>

The associated Fokker-Plank Eq'n is given by:

$$u_t + (\alpha(\varepsilon t)\, x\, u)_x = \frac{\beta^2(\varepsilon t)}{2} u_{xx} \ ; \quad u(x,0) = \delta(x)$$

or, w.r.t. slow variable $s = \varepsilon t$,

$$\varepsilon u_s + \alpha(s)\, x\, u_x + \alpha(s)\, u = \frac{\beta^2(s)}{2} u_{xx} .$$

We are interested in $\sigma^2(s) = \int^{\infty} x^2 u(x,s)\, dx$ [one can show that

Solving $\begin{cases} \varepsilon u_s + a(s)\, x u_x + d(s)\, u = \frac{\beta^2}{2} u_{xx} \\ u(x,0) = \delta(x) \end{cases}$ (F-P)

Change var ①: $\begin{cases} y = x e^{-\frac{A(s)}{\varepsilon}}, \text{ where } A(s) = \int_0^s a(s')ds' \\ u(x,s) = \mathcal{U}(y,s), \end{cases}$

$\rightsquigarrow \left.\begin{array}{l} u_x = e^{-A/\varepsilon}\,\mathcal{U}_y, \quad u_{xx} = e^{-2A/\varepsilon}\,\mathcal{U}_y \\ u_s = -x\frac{a e^{-A/\varepsilon}}{\varepsilon}\mathcal{U}_y + \mathcal{U}_s \end{array}\right\} \to \varepsilon u_s + a x u_x = \varepsilon \mathcal{U}_s$

$\Rightarrow \varepsilon \mathcal{U}_s + a(s)\,\mathcal{U} = \frac{\beta^2}{2} e^{-\frac{2A(s)}{\varepsilon}} \mathcal{U}_{yy}$

Change var ②: $\mathcal{U}(y,s) = e^{-\frac{A(s)}{\varepsilon}} V(y,s) \Rightarrow \varepsilon \mathcal{U}_s + a\mathcal{U} = e^{-\frac{A}{\varepsilon}} V_s$

$\Rightarrow \varepsilon V_s = \frac{\beta^2}{2} e^{-\frac{2A(s)}{\varepsilon}} V_{yy}$

Change var ③: $V(y,s) = W(y, \tau(s))$; Take $\varepsilon \tau'(s) = \frac{\beta^2}{2} e^{-\frac{2A(s)}{\varepsilon}}$

$\Rightarrow \boxed{W_\tau = W_{yy}}$; where $\boxed{\tau(s) = \frac{1}{2\varepsilon}\int_0^s e^{-\frac{2A(s')}{\varepsilon}} \beta(s')\, ds'}$

$\boxed{W(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{y^2}{4\tau}}}$

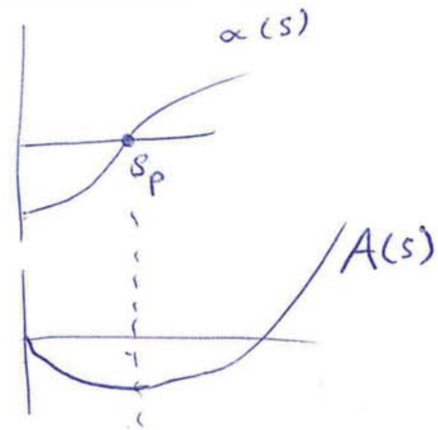
$$\Rightarrow \quad u(x,s) = \frac{e^{-x^2/(2\sigma^2)}}{\sqrt{\pi 2\sigma^2}} \quad , \text{ where}$$

$$\boxed{\sigma^2(s) = e^{\frac{2}{\varepsilon}A(s)} \int_0^s \beta^2(s')\, e^{-\frac{2}{\varepsilon}A(s')}\, ds'}$$

where $A(s) = \int_0^s \alpha(s)$

- Note that $A'(s_p) = 0$

and s_p is a min of $A(s)$



So apply Laplace's method:

If $s > s_p$ then

$$\sigma^2(s) \sim e^{\frac{2}{\varepsilon}A(s)} \beta^2(s_p)\, e^{-\frac{2}{\varepsilon}A(s_p)} \left(\frac{\pi}{\varepsilon A''(s_p)}\right)^{\frac{1}{2}}$$

$$\Rightarrow \boxed{\sigma(s) \sim \exp\left(\frac{1}{\varepsilon}\int_{s_p}^{s} \alpha(\tau)\right) \beta(s_p) \left(\frac{\pi}{\varepsilon\, \alpha'(s_p)}\right)^{\frac{1}{4}}}$$

Thus $u(s,x)$ is a Gaussian!

- Has mean zero;

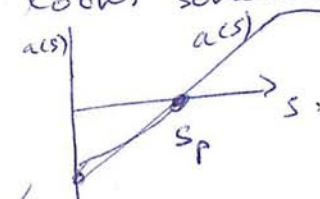
• <u>Compute variance</u>: $\sigma^2 = \int_{-\infty}^{\infty} u(x,s)\, x^2\, dx = e^{\frac{2A(s)}{\varepsilon}} \underbrace{\int V(y,s)\, y^2\, dy}_{\underbrace{\int W(y,\tau)\, y^2\, dy}_{2\tau(s)}}$

$u \to e^{-A/\varepsilon} V$, $\quad x = e^{A/\varepsilon} y$

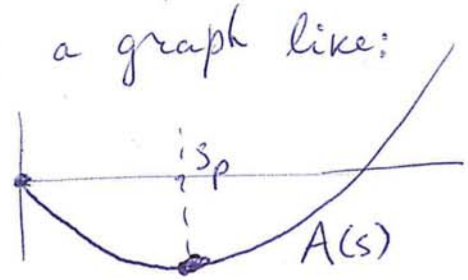
$$\Rightarrow \sigma^2 = e^{\frac{2A(s)}{\varepsilon}}\, 2\tau(s) = e^{\frac{2A(s)}{\varepsilon}}\, \frac{\sigma_0^2}{\varepsilon} \int_0^s e^{-\frac{2A(s')}{\varepsilon}}\, ds'$$

Next, apply <u>Laplace's method</u> to evaluate the resulting integral:

- Recall that $A(s) = \int_0^s a(s)\, ds$ and $a(s)$ looks something like:



so that $A(s)$ has a graph like:



- It has a <u>min</u> at $s = s_p$. Thus, for $s > s_p$, <u>we have</u>:

$$\int_0^s e^{-\frac{2A(s')}{\varepsilon}}\, ds' \sim e^{-2A(s_p)/\varepsilon} \int_{-\infty}^{\infty} e^{-A''(s_p)\frac{(s-s_p)^2}{\varepsilon}}\, ds$$

$$\sim e^{-\frac{2A(s_p)}{\varepsilon}} \sqrt{\frac{\varepsilon\pi}{A''(s_p)}}$$

(13)

- Define s_d to be such that $\sigma(s_d) \sim 1$; let $a_d = a(s_d)$.
  - Then $\begin{cases} \sigma(s) \ll 1 & \text{if } s < s_d \text{ and} \\ \sigma(s) \gg & \text{if } s > s_d \end{cases}$

So s_d satisfies:

$$\ast \quad \boxed{\begin{gathered} \alpha(s_p) = 0 \\ \int_{s_p}^{s_d} \alpha(s)\,ds + \varepsilon \ln \left\{ \beta(s_p) \left(\frac{\pi}{\varepsilon \alpha'(s_p)} \right)^{\frac{1}{4}} \right\} = 0 \end{gathered}}$$

with $\boxed{\alpha_d = \alpha(s_d).}$

- Back to $\alpha(s) = -1 + s$, $\beta(s) = \beta$, then $s_p = 1$,

and $\boxed{a_d = \sqrt{-2\varepsilon \ln\left(\beta \left(\frac{\pi}{\varepsilon}\right)^{\frac{1}{4}}\right)}}$

Ex. $\varepsilon = 0.02$, $\beta = 10^{-5}$ $\Rightarrow$ $\boxed{a_d = 0.64}$ Excellent agreement with theory!

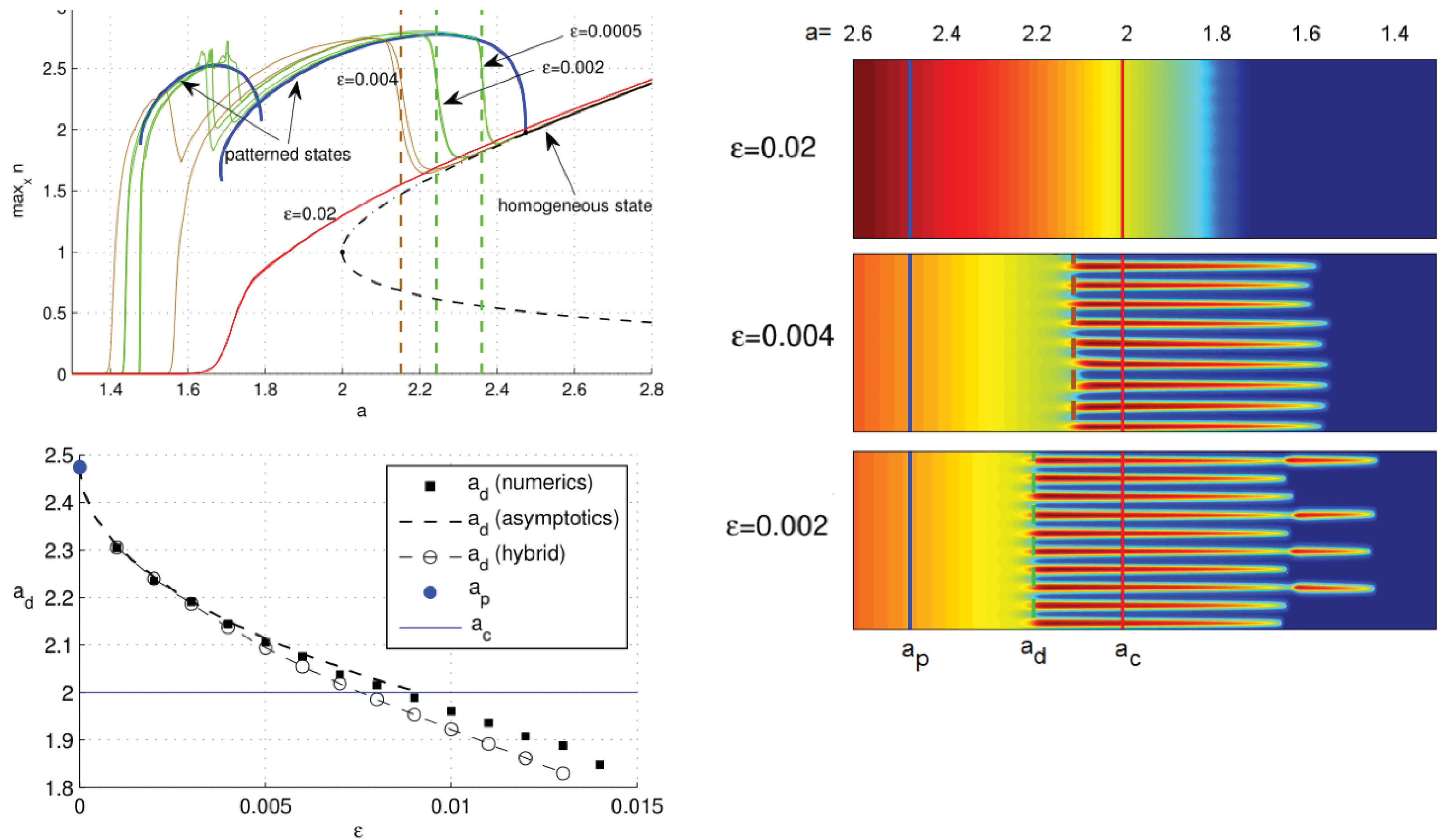


FIG. 2. Numerical verification of a_d as given in (16) for (6). The slow drift is taken to be $a = a_0 - \varepsilon t$ with $a_0 = 3$, $\sigma_0 = 0.0001, L = 22.839$ and $\delta = 0.05$. Top left: the evolution of max n as a function of a with ε as indicated. The dashed lines show a_d as given by (16). The patterned state (blue on-line) to the left corresponds to a single-bump solution, whereas the patterned state on the right corresponds to the wave-number 10 born from a Turing bifurcation at $a = a_p = 2.47439$. Top right: color plot of $n(x,t)$ as it evolves in time. The Turing bifurcation point a_p, the delayed bifurcation a_d and the fold point $a_c = 2$ are indicated. Bottom left: Comparison of asymptotic and full numerical results for a_d. The hybrid curve is obtained by using the full homogeneous state (17) instead of (7) when computing (16). The value of a_d is estimated numerically as discussed in the text, and an average over 50 simulations is used. We used $N = 100, dt = 0.1$ (see Appendix A for numerical implementation details).