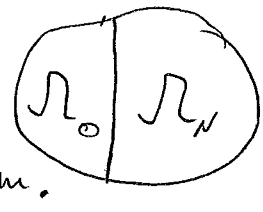
Patch models in ecology

Q: Is organic agriculture sustainable?

- In conventional agriculture, pathogens [such as fungi, arthropods, bacteria...] are controlled using pesticides
- Pesticides are not used in organic farming.
- If organic farm is small and next to a conventional farm, pathogens may still be suppressed, due to effect of diffusion.
- However, if organic farm is too big, it may become a source of infestation.

Mathematical modelling:

- Assume the farm Ω is divided into two regions: $\overline{\Omega} = \overline{\Omega}_0 \cup \overline{\Omega}_N$ representing organic and non-organic farm.



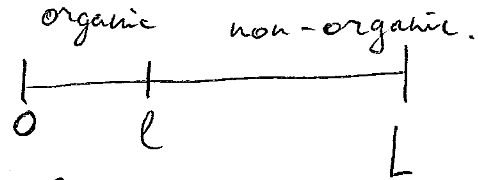
- Pathogen spreads trough diffusion
- Pathogen grows logistically inside Ω_0
- Inside Ω_N, the growth of pathogen is suppressed using pesticides

Model:
$$\begin{cases} u_t = D\Delta u + g(x)u - u^2, & x \in \Omega \\ \partial_n u = 0, & x \in \partial\Omega \end{cases} \qquad (1)$$

where
$$g(x) = \begin{cases} > 0, & x \in \Omega_0 \\ < 0, & x \in \Omega_N \end{cases} \quad \text{(growth rate)} \qquad (2)$$

First, consider the zero-state $u \equiv 0$.
If we linearize around it, we get the problem
$$\begin{cases} \lambda\varphi = D\Delta\varphi + g(x)\varphi, & x \in \Omega \\ \partial_n\varphi = 0, & x \in \partial\Omega. \end{cases} \qquad (3)$$

To get some insight, consider a <u>one-dimensional case</u>:
$$\bar\Omega = [0, L], \quad \bar\Omega_0 = [0, \ell], \quad \bar\Omega_N = [\ell, L]$$



Also let's take $D = 1$, $\quad g(x) = \begin{cases} +a, & x \in [0, \ell) \\ -b, & x \in (\ell, L] \end{cases}$

Then (3) becomes
$$\begin{cases} \lambda\varphi = \varphi'' + a\varphi, & x \in (0, \ell) \\ \lambda\varphi = \varphi'' - b\varphi, & x \in (\ell, L) \\ \varphi'(0) = 0 = \varphi'(L) \\ \varphi(\ell^-) = \varphi(\ell^+), \quad \varphi'(\ell^-) = \varphi'(\ell^+) \end{cases}$$

On $(0,l)$: $\quad \varphi = A\cos(\sqrt{a-\lambda}\,x)$,

On (l,L): $\quad \varphi = B\cosh(\sqrt{b+\lambda}\,(x-L))$

By scaling, take $A=1$; continuity conditions:

$$\cos(\sqrt{a-\lambda}\,l) = B\cosh(\sqrt{b+\lambda}\,(l-L))$$

$$-\sqrt{a-\lambda}\,\sin(\sqrt{a-\lambda}\,l) = B\sqrt{b+\lambda}\,\sinh(\sqrt{b+\lambda}\,(l-L))$$

$$(*) \quad \Rightarrow \quad \boxed{-\sqrt{a-\lambda}\,\tan(\sqrt{a-\lambda}\,l) = \sqrt{b+\lambda}\,\tanh(\sqrt{b+\lambda}\,(l-L))}$$

__Q__: For given parameter values, is $\lambda<0$?

- Note that $\lambda<0 \Rightarrow$ stability [no infestation]

  $\lambda>0 \Rightarrow$ instability of zero state [outbreak occurs]

- Threshold happens if $\lambda=0$

  $$\Rightarrow \sqrt{a}\,\tan(\sqrt{a}\,l) = \sqrt{b}\,\tanh(\sqrt{b}\,(L-l))$$

  If $L \gg 1 \Rightarrow \sqrt{a}\tan\sqrt{a}\,l = \sqrt{b}$

  $$\Rightarrow \boxed{l_c = \frac{1}{\sqrt{a}}\arctan\left(\frac{\sqrt{b}}{\sqrt{a}}\right)}$$

- Can we show that $\lambda<0$ if $l<l_c$

  $\lambda>0$ if $l>l_c$?

- Use Rayleigh-Ritz formulation: the biggest eigenvalue of (3) satisfies:

$$(4) \qquad \lambda = \max_{\varphi \in C^1(\Omega)} \frac{\int_\Omega -D|\nabla\varphi|^2 + g(x)\,\varphi^2}{\int_\Omega \varphi^2}$$

- Now Suppose that there is no organic plot, i.e. $g(x) < 0 \quad \forall x \in \Omega$. Then $\frac{\int -D|\nabla\varphi|^2 + g(x)\varphi^2}{\int \varphi^2} \leq 0 \ \forall \varphi$

  $\Rightarrow \boxed{\lambda < 0}$

  $\Rightarrow$ no outbreak if no organic farm (as expected)

- On the other extreme, suppose the whole farm is organic, i.e. $g(x) > 0 \ \forall x \in \Omega$. Then use $\varphi = 1$ as a test function in (4) to conclude that $\lambda \geq \text{avg}(g) > 0$

  $\Rightarrow$ outbreak will occur if the average growth rate of pathogen is positive (in particular, if organic farm is sufficiently large).

- From (4) it also follows that increasing $g(x)$ will increase λ, i.e. increasing the size of organic farm will increase the chance of outbreak.

Non-zero steady states

Thm [Berestyki et.al, 2004]:

Consider the model (1) and its steady state:

$$(5) \quad \begin{cases} 0 = D\Delta u + g(x)u - u^2, & x \in \Omega \\ \partial_n u = 0, & x \in \partial\Omega \end{cases}.$$

Let λ_0 be the biggest eigenvalue of the zero solution, i.e.

$$(6) \quad \begin{cases} \lambda_0 \varphi_0 = D\Delta\varphi_0 + g(x)\varphi_0, & x \in \Omega \\ \partial_n \varphi_0 = 0, & x \in \partial\Omega \end{cases}.$$

1. Suppose $\lambda_0 < 0$. Then the only sol'n to (5) is the zero solution $u=0$, and it is stable.
2. Suppose $\lambda_0 > 0$. Then there exists a unique positive sol'n to (5), $u(x) > 0 \ \forall x \in \Omega$.

   Moreover, this solution is asymptotically stable, i.e. the corresponding eigenvalue pblm,

$$(7) \quad \begin{cases} \lambda\varphi = D\Delta\varphi + g(x)\varphi - 2u\varphi, & x \in \Omega \\ \partial_n \varphi = 0, \quad x \in \partial\Omega \end{cases}$$

   admits only negative eigenvalues, $\lambda < 0$.
3. Suppose that $\Omega = B_R(0) = \{x : |x| < R\}$ and that $g(x) = g(|x|)$ and g is decreasing in $|x|$. Also suppose that $\lambda_0 > 0$ as in case 2. Then the positive solution is radially symmetric and decreasing, i.e. $u(x) = u(|x|)$ and $u'(|x|) \leq 0$.

Proof: To show 1., multiply (5) by φ_0

$$\int_\Omega D\,\varphi_0(\Delta u + g u - u^2) = 0$$

Integrate by parts: $\int_\Omega \varphi_0 \Delta u = \int_\Omega \Delta\varphi_0 u + \underbrace{\int_{\partial\Omega} \partial_n u\,\varphi_0 - \partial_n \varphi_0 u}_{0}$

$$\Rightarrow \int_\Omega u\,(\underbrace{D\Delta\varphi_0 + g\varphi_0}_{\lambda\varphi}) = \int_\Omega u^2 \varphi_0$$

$$(*) \quad \lambda\int_\Omega u\varphi_0 = \int_\Omega u^2\varphi_0$$

Now if φ_0 is the eigenfunction corresponding to the biggest eigenvalue λ_0, then $\varphi_0 \neq 0$

But then $(*)$ is only possible if $\lambda_0 > 0$ or else $u \equiv 0$ ■

step2:
Next to show existence of $u > 0$ in case $\lambda > 0$:
we use sub-super solutions. Let $\overline{u} = \max_{x\in\Omega} g(x)$.

Then $D\Delta\overline{u} + g\,\overline{u} - \overline{u}^2 = (g - \overline{u})\,\overline{u} \leq 0$

So $\overline{u}$ is a supersolution of (5).

The subsolution is given by $\underline{u}(x) = \alpha\,\varphi(x)$

where α is to be chosen;

Then $D\Delta \underline{u} + g\,\underline{u} - \underline{u}^2 = \alpha \lambda \varphi_0 - \alpha^2 \varphi_0^2$
$$= \alpha \varphi_0 (\lambda - \alpha \varphi_0)$$

Since $\lambda_0 > 0$ and WLOG, $\varphi_0 > 0$ inside Ω,
$\underline{u}$ is subsolution provided that $\alpha \leq \dfrac{\lambda_0}{\max_\Omega \varphi_0}$.

Moreover we can make $\underline{u} \leq \overline{u}$ by choosing $\alpha \leq \dfrac{\overline{u}}{\max \varphi_0}$ so we <u>choose</u>:

$$\alpha = \min\left(\frac{\lambda_0}{\max \varphi_0}, \frac{\overline{u}}{\max \varphi_0}\right).$$

Then $\underline{u} \leq \overline{u}$ and the method of subsolutions / supersolutions [see for ex. Evans] guarantees the existence of a solution to (5) with $\underline{u}(x) \leq u(x) \leq \overline{u}$. Also $u(x) > 0$ inside Ω since $\varphi_0(x)$ is.

<u>Step 3:</u> We prove stability of positive sol'n.
Consider a sol'n $u(x) > 0$ of (5) and let (λ_0, φ_0) be the sol'n to (6) with $\lambda_0 > 0$, $\varphi_0 > 0$
[By step **1**, such solution exists]
Now let (λ, φ) be the biggest eigenvalue sol'n to (7), so that $\varphi > 0$ inside Ω. Multiply (7) by φ_0 & integrate by parts:

$$\lambda \int_\Omega \varphi \varphi_0 = \int_\Omega \varphi \varphi_0 (\lambda_0 - 2u) \tag{8}$$

Now from <u>step 2</u>, note that

$$u \geq \left(\frac{1}{\max_{\Omega} \varphi_0}\right) \min\left(\lambda_0, \max_{\Omega} g\right) \varphi_0(x)$$

$$\geq \min\left(\lambda_0, \max_{\Omega} g\right)$$

Moreover,

$$\lambda_0 = \max_{\varphi_0 \in C^1(\Omega)} \underbrace{\frac{\int_{\Omega} -D|\nabla \varphi_0|^2 + g(x)\varphi_0^2}{\int_{\Omega} \varphi_0^2}}_{\leq \frac{\int g \varphi^2}{\int \varphi^2} \leq \max_{\Omega} g}$$

$$\Rightarrow \lambda_0 \leq \max g$$

$$\Rightarrow \quad u(x) \geq \lambda_0$$

Thus (8) becomes: $\quad \lambda \int \varphi \varphi_0 \leq -\lambda_0 \int \varphi \varphi_0$

$$\Rightarrow \quad \lambda \leq -\lambda_0 < 0$$

This proves that all (real) eigenvalues of (7) are negative
But (7) is self-adjoint $\Rightarrow$ all positive solutions are <u>stable</u>.

<u>Step 4</u>: <u>Uniqueness</u>: The basic idea is that stability should somehow imply uniqueness.

Assume we have two solutions, $u_1, u_2 > 0$

If $u_1 \neq u_2$ inside Ω then $\exists$ a point x s.t.

WLOG $u_2(x) > u_1(x)$. Define

$$D = \{x : u_2(x) > u_1(x),\ x \in \Omega\};\ D \neq \emptyset.$$

Let $w(x) = u_2(x) - u_1(x)$, so that

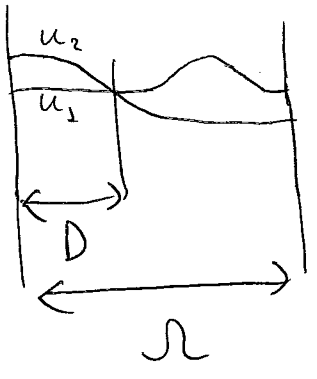
$$(9) \qquad \Delta w + g w - (u_1 + u_2) w = 0$$

Now let φ_2, λ_2 be the principal eigenvalue / eigenvector pair sol'n to (7) with $u = u_2$; so that $\varphi_2 > 0$ inside Ω and $\lambda_2 < 0$ by step 3. Multiply (9) by φ_1 & integrate on D by parts:

$$0 = \int_{\partial D} (\varphi_1 \partial_n w - w \partial_n \varphi_1) + \int_D w(\underbrace{\Delta\varphi_1 + g\varphi_1}_{\lambda_1\varphi_1 + 2u_2\varphi_1}) - \int_D (u_1 + u_2) w \varphi_1$$

Now note that if $x \in \partial D$ then either $w(x) = 0$ or else $x \in \partial\Omega$

In either case, $\partial_n w \leq 0$ and $w\,\partial_n \varphi_1 = 0$



so that the boundary term ≤ 0.

$$\Rightarrow \int_D \lambda_1 \varphi_1 w + \int_D w \varphi_1 (\underbrace{2u_1 - (u_1 + u_2)}_{-w}) \geq 0$$

$$\Rightarrow \int_D \varphi_2 w (\lambda_1 - w) \geq 0,$$ contradicting the fact that $\lambda_1 < 0, w > 0, \varphi_2 > 0$.

Step 5: Radial symmetry: If $\Omega = B_R(0)$, $g(x) = g(|x|)$:

First, note that $u(x)$ must be radially symmetric; otherwise rotating it yields another solution which would violate uniqueness.

So $u(x) = u(r)$, $r = |x|$ satisfies:

$$\begin{cases} u_{rr} + \frac{n-1}{r} u_r + gu - u^2 = 0 & (10a) \\ u'(0) = 0, \quad u'(R) = 0 & (10b) \end{cases}$$

Then

$$u_{rrr} + \frac{n-1}{r} u_{rr} - \frac{n-1}{r^2} u_r + g'(r) u + (g - 2u) u_r = 0 \tag{11}$$

Now suppose that $u'(r) > 0$ for some $r \in (0, R)$.

Let r_0 be the point where $u'(r)$ has its max. Then $r_0 \in (0, R)$ because of (10b)

Thus $u'(r_0) > 0$; $u''(r_0) = 0$, $u'''(r_0) \leq 0$

So from (10a) we get $g(r_0) - u(r_0) \leq 0$;

from (11) we get: $g(r_0) - 2u(r_0) \geq 0$

$\Rightarrow u(r_0) \leq 0$

But this contradicts the assumption $u > 0$. ∎

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[for a proof of sub-supersolution method]

Some Questions

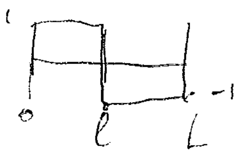
1) What is the optimal location for organic farm?

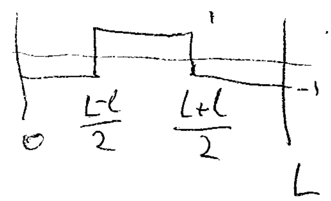
a) Consider the problem

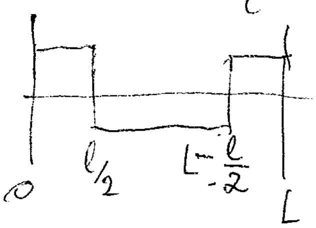
$$\lambda \varphi = \varphi'' + g(x)\varphi, \quad \varphi'(0) = 0 = \varphi'(L) \qquad (Q1)$$

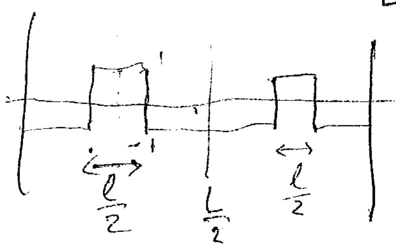
where
$$g(x) = \begin{cases} 1, & x \in (a, a+\ell) \\ -1, & x \in [0, L] \setminus (a, a+\ell) \end{cases}$$

Show that λ is minimized when $a = \frac{L-\ell}{2}$, i.e. when the organic farm is located in the middle of the interval.

b) Given $g_1 =$ 

$g_2 =$ 

$g_3 =$ 

$g_4 =$ 

Which configuration minimizes λ?

2) Write out the details for using sub-supersolution method to show existence of non-constant (outbreak) sol'n. of (5).

3) Consider equation (5) in the limit $D \gg 1$.

a) Show (formally) that the outbreak occurs iff $\int_\Omega g(x) > 0$ as $D \to \infty$.

b) Show that the outbreak sol'n is given by

$$u(x) \sim \bar{g} = \frac{\int_\Omega g}{\text{area}(\Omega)} + O\left(\frac{1}{D}\right), \quad D \to \infty.$$

c) What can you say about (5) in the limit $D \ll 1$?

d) If we replace u^2 by $f(u)$ in (5), under what conditions on $f(u)$ is the theorem still valid?