Discrete Fourier transform

Suppose we are given $f(t)$, a continuous, periodic function that has period 2π.

Then
$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$$
where
$$a_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt .$$

Now imagine that $f(t)$ is a signal [e.g. Audio] that is being sampled at a rate Δt;
that is, we only know $f_n = f(t_n)$
where $t_n = \Delta t\, n$

Suppose moreover that
$$\Delta t = \frac{2\pi}{N}$$
[i.e. we sample N times in a single period].

Then we can use Reimann approximation
$$a_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \approx \frac{1}{2\pi} \sum_{n=0}^{N-1} \Delta t\, f_n e^{-ikt_n}$$

<u>Question</u>: How well is a_k approximated?

Let
$$\hat{a}_k = \frac{1}{2\pi} \sum_{n=0}^{N-1} \Delta t \, f_n \, e^{-ikt_n}$$
$$\hat{a}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{-i\frac{2\pi k n}{N}}$$

- When is $\hat{a}_k = a_k$?
- First, suppose that $f(t) = e^{imt}$ [pure waves, single frequency]

Then
$$\hat{a}_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}(m-k)n}.$$

- If $m-k \equiv 0 \pmod N$ then
$$e^{2\pi\frac{(m-k)}{N}i} = 1, \text{ so that}$$
$$\hat{a}_k = \frac{1}{N}\sum_0^{N-1} 1 = 1$$

- If $m-k \not\equiv 0 \pmod N$ then
$$\sum e^{i\frac{2\pi(m-k)}{N}n} = \sum_0^{N-1} z^n \quad [z = e^{2\pi i\frac{(m-k)}{N}}]$$
$$= \frac{z^N - 1}{z - 1} = 0$$

<u>Thus:</u>
$$\hat{a}_k = \begin{cases} 1, & \text{if } m \equiv k \pmod N \\ 0, & \text{otherwise} \end{cases}$$

[if $f(t) = 1 \cdot e^{imt}$]

In particular, if $|m| < \frac{N}{2}$ then

$$\hat{a}_k = a_k \quad \forall \ |k| < \frac{N}{2}$$

Similarly, suppose

$$f(t) = \sum a_m e^{imt}.$$ Then

$$\hat{a}_k = \sum_{m \equiv k \pmod{N}} a_m$$

In particular, if

$$f(t) = \sum_{m=0}^{N-1} e^{imt} a_m, \quad f_n = \sum_{m=0}^{N-1} e^{im\frac{2\pi n}{N}} a_m$$

then $\hat{a}_k = a_k$ so that $f_n = \sum \hat{a}_n e^{im\frac{2\pi \cancel{M}}{N}}$

This relationship is called "Discrete Fourier Transform."

To summarize:

$$\boxed{\begin{aligned} f_n &= \sum_{m=0}^{N-1} \hat{a}_m e^{\frac{2\pi i n m}{N}}, \quad n = 0 \ldots N-1 \\ &\text{if and only if} \\ \hat{a}_m &= \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-\frac{2\pi m n i}{N}}, \quad m = 0 \ldots N-1 \end{aligned}}$$

Remark: in Matlab, if f is an array $N \times 1$ then

$$\hat{a} = fft(f) \quad \text{is its D.F.T.}$$

[fft stands for "fast fourier tranform]

and and conversely, $f = ifft(\hat{a})$

Fast Fourier Transform (FFT):

- Computing all $\hat{a}_m$ from (*) takes $O(N^2)$ time
  [i.e. $O(N)$ for each a_m, $m = 0 \dots N-1$]
- FFT is a way to compute all of $\hat{a}_m$ in $O(N \ln N)$ time.

- Let $z_m = N \hat{a}_m$; let $\omega_N = e^{-\frac{2\pi i}{N}}$.

e.g. if $N = 4$, we have:

$$z_m = \omega_4^{m\cdot 0} f_0 + \omega_4^{m\cdot 1} f_1 + \omega_4^{m\cdot 2} f_2 \quad \omega_4^{m\cdot 3} f_3$$

$$= \omega_4^0 \left(f_0 + \omega_4^{2m} f_2\right) + \omega_4^1 \left(f_1 + \omega_4^{2m} f_3\right)$$

$$= \underbrace{\left(f_0 + \omega_2^m f_2\right)}_{\text{FFT w. } N=2} + \omega_4^1 \underbrace{\left(f_1 + \omega_2^m f_3\right)}_{\text{FFT w. } N=2}$$

If $N=8$:

$$z_m = \omega_8^{0\cdot m}\left(f_0 + \omega_8^{2m} f_2 + \omega_8^{4m} f_4 + \omega_8^{6m} f_6\right)$$

$$+ \omega_8^{1\cdot m}\left(f_1 + \omega_8^{2m} f_3 + \omega_8^{4m} f_5 + \omega_8^{4m} f_7\right)$$

$$= \underbrace{\left(f_0 + \omega_4^{m} f_2 + \omega_4^{2m} f_4 + \omega_4^{3m} f_6\right)}_{\text{FFT, } N=4} + \omega_8^{1}\underbrace{\left(f_1 + \omega_4^{2m} f_3 + \omega_4^{4m} f_5 + \omega_4^{6\ldots}\right.}_{\text{FFT, } N=4}$$

Conclusion:

Let $T(N) \equiv$ time for FFT_N;

then $T(2N) = T(N)\cdot 2 + K$

$$\Rightarrow \boxed{T(N) = O(N \log N)}$$

Gibbs Phenomenon and convergence of Fourier series

Suppose $f(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt}$ can be (1)

represented by Fourier series where

$$a_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ims} f(s)\, ds \quad (2)$$

Consider a partial sum:

$$f_N(t) \stackrel{def}{=} \sum_{-N}^{N} a_m e^{imt} \quad (3)$$

The statement "$f(t)$ can be represented by (1, 2)" means "$f_N(t) \to f(t)$ as $N \to \infty$."

The fundamental theorem for Fourier series is that this is the case provided that $f(t)$ is continuous and 2π-periodic on $\mathbb{R}$

Key idea: Plug (2) into (3):

$$f_N(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-N}^{N} e^{im(t-s)} f(s)\, ds$$

The sum can be summed up explicitly:

$$\text{Let } K(\varphi) = \sum_{-N}^{N} e^{im\varphi}$$

$$= e^{-iN\varphi}\left(1 + e^{i\varphi} + \dots + e^{i2N\varphi}\right)$$

$$= e^{-iN\varphi}\left(\frac{1-e^{i\varphi\, 2N+1}}{1-e^{i\varphi}}\right) = \frac{e^{-iN\varphi} - e^{i\varphi(N+1)}}{1-e^{i\varphi}}$$

$$= \frac{e^{-i(N+\frac{1}{2})\varphi} - e^{i\varphi(N+\frac{1}{2})}}{e^{-i\varphi/2} - e^{i\varphi/2}}$$

$$= \frac{\sin\left((N+\frac{1}{2})\varphi\right)}{\sin\left(\frac{\varphi}{2}\right)}$$

<u>So we get:</u>

$$\boxed{f_N(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(t-s) f(s)\, ds \quad \text{where} \quad K(\varphi) = \frac{\sin\left((N+\frac{1}{2})\varphi\right)}{\sin\left(\frac{\varphi}{2}\right)}}$$

<u>Key idea:</u> Let $\varepsilon = \frac{1}{N+\frac{1}{2}}$; write

$$f_N(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(\varphi) f(t-\varphi)\, d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(\frac{\varphi}{\varepsilon}\right)}{\sin(\varphi/2)} f(t-\varphi)\, d\varphi$$

$u = \frac{\varphi}{\varepsilon}$

$$\boxed{f_N(t) = \frac{1}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{\sin(u)}{\sin\left(\frac{\varepsilon u}{2}\right)}\, \varepsilon\, f(t-\varepsilon u)\, du, \quad \varepsilon = \left(N+\tfrac{1}{2}\right)^{-1}}$$

Next, estimate: ① replace $\int_{-\pi/\varepsilon}^{\pi/\varepsilon}$ by $\int_{-\infty}^{\infty}$

② replace $\sin(\varepsilon \frac{u}{2}) \sim \varepsilon \frac{u}{2}$

Then formally, we get:

$$f_N(t) \sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} f(t - \varepsilon u)\, du$$

③

Finally, take limit $\varepsilon \to 0$, $f(t-\varepsilon u) \to f(t)$

so that $f_N(t) \sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} f(t)\, du$

$$\sim \frac{f(t)2}{\pi} \int_0^{\infty} \frac{\sin u}{u}\, du$$

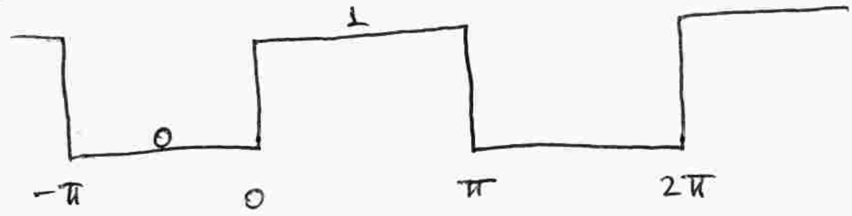
- One can show that $\int_0^{\infty} \frac{\sin u}{u}\, du = \frac{\pi}{2}$

so we get $f_N(t) \to f(t)$ as $N \to \infty$

- Steps ① and ② can be made rigorous for any integrable $f(t)$
- step ③ can be made rigorous for any continuous $f(t)$

Gibbs Phenomenon:

Suppose $f(t) =$



is a square wave, $f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases}$.

Compute:
$$a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-imt}\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} e^{-imt} = \begin{cases} \frac{1}{2} & \text{if } m = 0 \\ \frac{1}{2\pi i m}\left(1 - e^{-im\pi}\right) \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{if } m = 0 \\ 0 & \text{if } m \text{ is even} \\ \frac{1}{im\pi} & \text{if } m \text{ is odd} \end{cases}$$

Thus formally we write,

$$f(t) = \sum_{-\infty}^{\infty} a_m e^{imt} = \frac{1}{2} + \frac{1}{i\pi}\left(e^{it} - e^{-it} + \frac{e^{3it}}{3} - \frac{e^{-3it}}{3} + \dots\right.$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots\right)$$

$$\text{Let } f_N(t) = \sum_{-N}^{N} a_m e^{imt} = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \dots \frac{1}{N}\sin Nt\right)$$

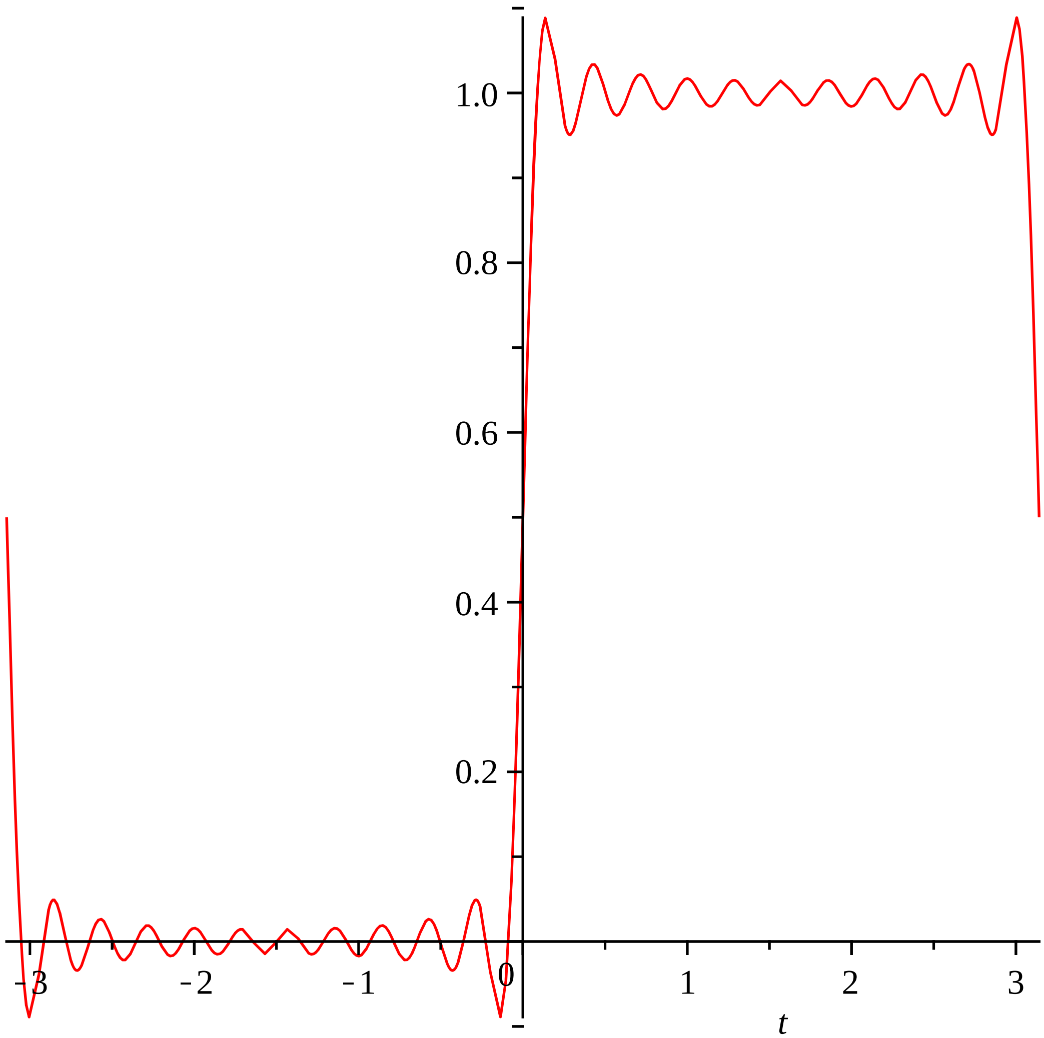
[N odd]

be the partial sum.

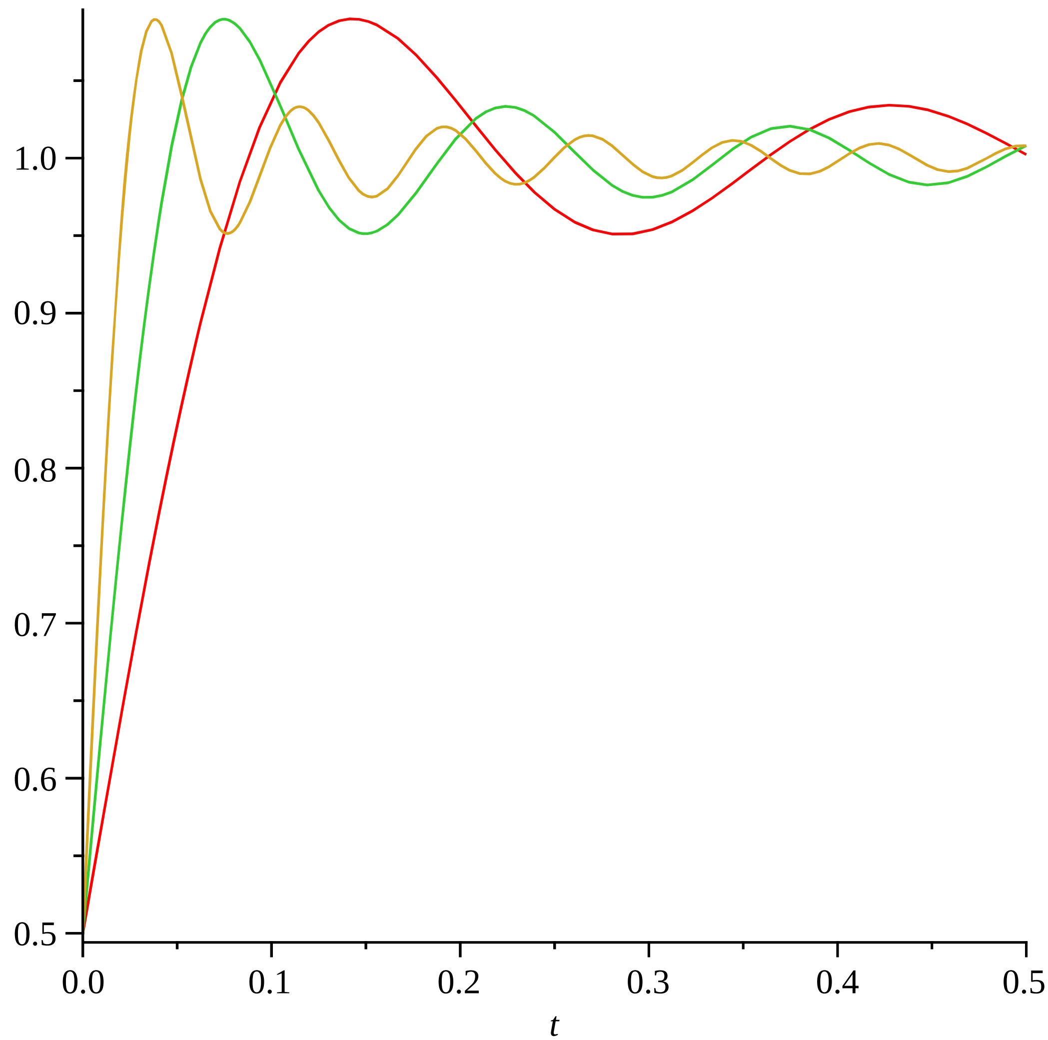
The figure on next page ~~shows~~ compares f, f_{20}, f_{40}, f_{8}

. Note that it appears that $f_N(t) \to f(t)$ ~~ex~~ except near $t \sim 0$;

```
> f20 := 2/Pi*sum(sin((2*n+1)*t)/(2*n+1),n=0..10)+1/2:
> f40 := 2/Pi*sum(sin((2*n+1)*t)/(2*n+1),n=0..20)+1/2:
> f80 := 2/Pi*sum(sin((2*n+1)*t)/(2*n+1),n=0..40)+1/2:
> plot(f20,t=-Pi..Pi);
```




```
>
> plot([f20,f40,f80], t=0..0.5);
```



1.0
0.9
0.8
0.7
0.6
0.5
0.0
0.1
0.2
0.3
0.4
0.5
t

Near $t \approx 0$, f_N has a max of about 1.09, an <u>overshoot</u> of about 9%, which appears to persist even as N is increased.

This is precisely the Gibbs phenomenon: f_N oscillates highly and overshoots $f(t)$ by about 9% near a jump discontinuity of $f(t)$.

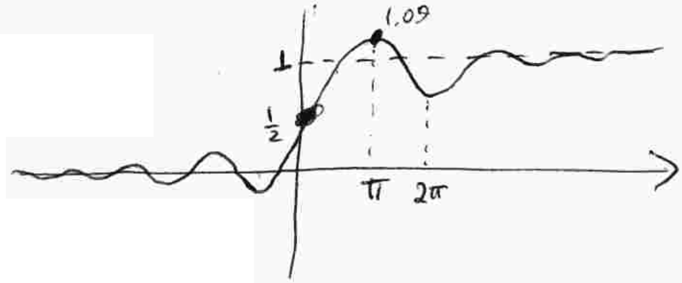
To explain this, recall we approximated

$$f_N(t) \sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} \underbrace{f(t - \varepsilon u)}_{\begin{cases} 0 & \text{if } t - \varepsilon u < 0 \Leftrightarrow u > \frac{t}{\varepsilon} \\ 1 & \text{if } u < t/\varepsilon \end{cases}} \, du \quad \left[\text{where } \varepsilon = \frac{1}{N + \frac{1}{2}} \right]$$

$$\sim \frac{1}{\pi} \int_{-\infty}^{t/\varepsilon} \frac{\sin u}{u} \, du \sim F\left(\tfrac{t}{\varepsilon}\right)$$

~~Suppose~~ where $F(z) = \frac{1}{\pi} \int_{-\infty}^{z} \frac{\sin u}{u} \, du$

Now $F(z)$ is the so-called Frenzel Integral; its graph is:



- $F'(z) = \frac{1}{\pi} \frac{\sin z}{z} = 0$ iff $z = \pm\pi, \pm 2\pi, \dots$

- Its <u>max</u> occurs at $z = \pi$;

we evaluate : $F(\pi) = \frac{1}{2\pi}\left(\int_{-\infty}^{0} + \int_{0}^{\pi}\right)\frac{\sin u}{u}$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_0^{\pi}\frac{\sin u}{u}\,du \approx 1.09$$

This expalins Gibbs oscillations / phenomenon.

The following figure compares

$F\left(\frac{t}{\varepsilon}\right)$ vs. f_N [recall $\varepsilon = \frac{1}{N+\frac{1}{2}}$]

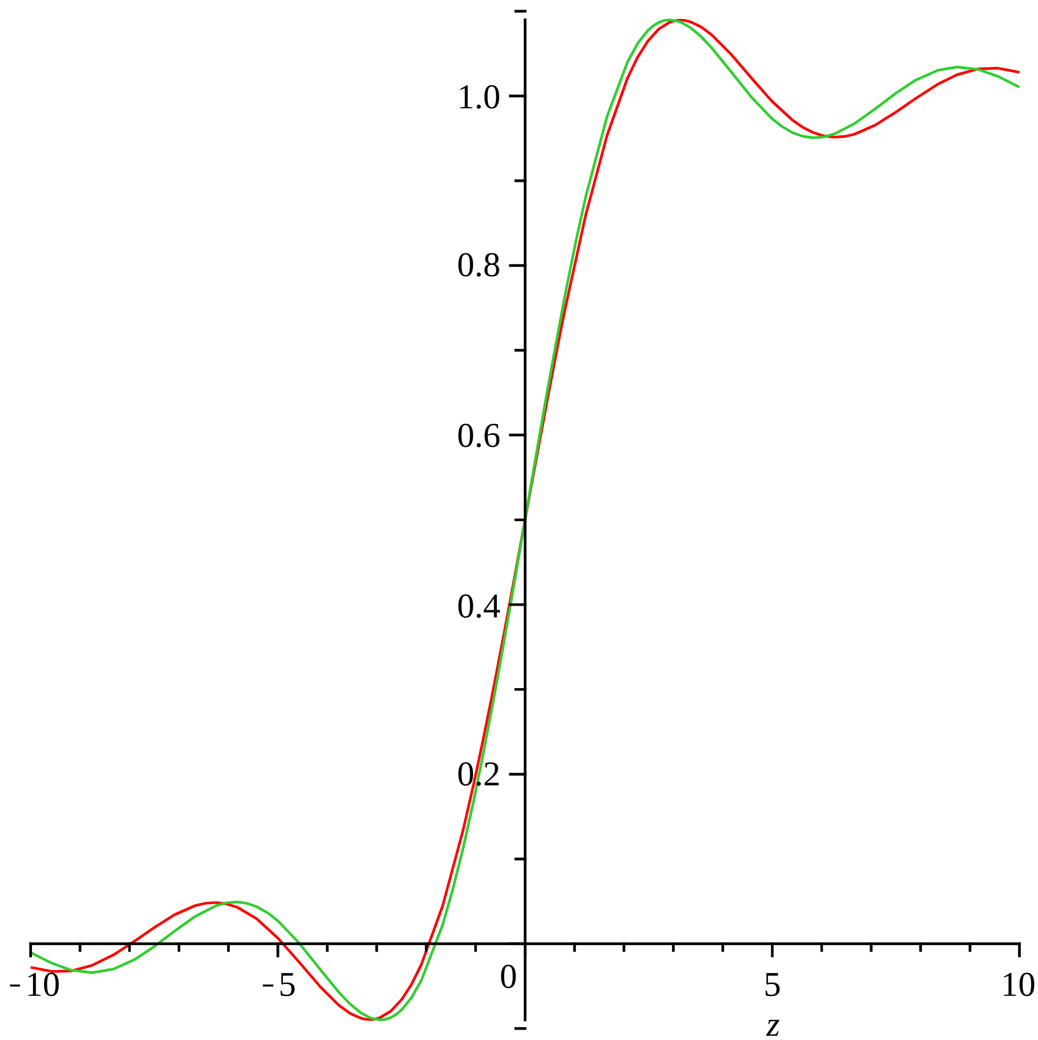
for several values of N.

Note the excellent agreement.

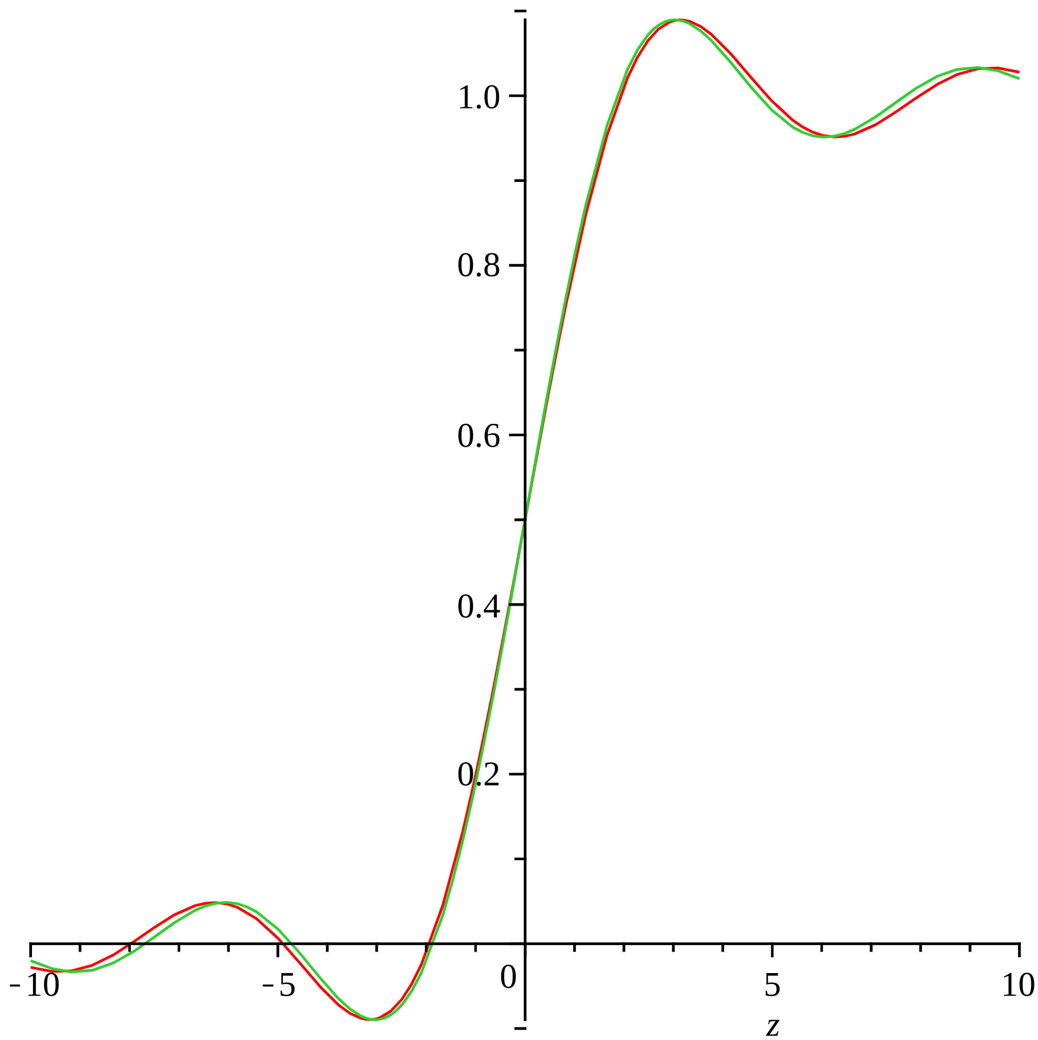
```
> F := 1/Pi*int(sin(u)/u,u=-infinity..z);
```

$$F := \frac{\frac{1}{2}\pi + \mathrm{Si}(z)}{\pi} \tag{1}$$

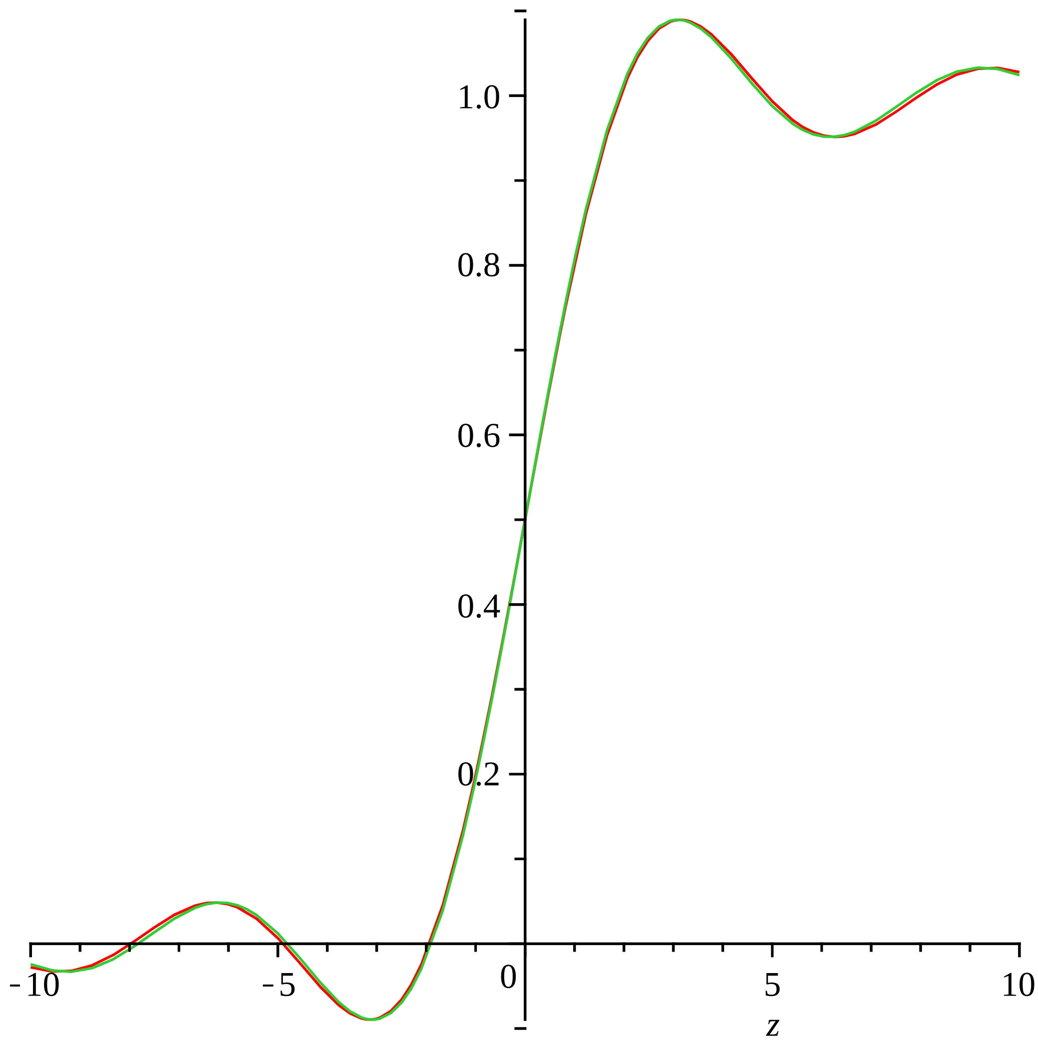
```
> plot([F,eval(f20,t=z/20.5)],z=-10..10);
```




```
> plot([F,eval(f40,t=z/40.5)],z=-10..10);
```




```
> plot([F,eval(f80,t=z/80.5)],z=-10..10);
```


1.0
0.8
0.6
0.4
0.2
-10
-5
0
5
10
z

>