Spike patterns and their stability

Consider the Gierer-Meinhardt model:

$$(GM)\begin{cases} a_t = \varepsilon^2 \Delta a - a + \dfrac{a^2}{h}, & x \in \Omega \\ \tau h_t = D\Delta h - h + a^2 \\ \partial_n h = 0 = \partial_n a & \text{on } \partial\Omega. \end{cases}$$

Suppose that $D \gg \varepsilon^2$.

Let x_0 be a max of a, and consider the steady-state for $\Omega = [-L, L] \in \mathbb{R}$:

$$\begin{cases} \varepsilon^2 a_{xx} - a + \dfrac{a^2}{h} = 0 \\ D h_{xx} - h + a^2 = 0 \\ h'(\pm L) = a'(\pm L) = 0 \end{cases}, \quad x \in [-L, L]$$

Near $x = x_0$, we rescale:

$$x = x_0 + \varepsilon y \iff y = \frac{x - x_0}{\varepsilon};$$

$$a(x) = A(y), \quad h(x) = H(y).$$

$$\Rightarrow \begin{cases} A_{yy} - A + \dfrac{A^2}{H} = 0 \\ \dfrac{D}{\varepsilon^2} H_{yy} - H + A^2 = 0 \end{cases}$$

Thus to leading order, ~~H(y)~~ H_0 is a constant, and we get

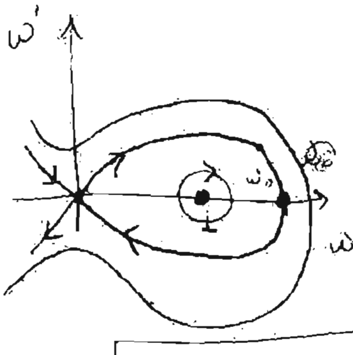
$$A_{yy} - A + \frac{A^2}{H_0} = 0.$$

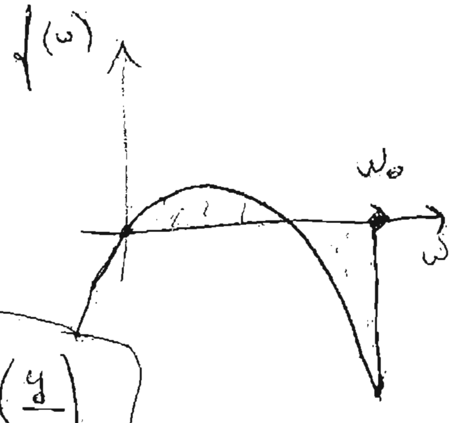
Next, <u>rescale</u>: $A = \omega H_0$; then

$$\begin{cases} \omega_{yy} = \omega - \omega^2 = f(\omega) \\ \omega_y \to 0 \quad \text{as } y \to \pm\infty, \end{cases}$$

$$\omega'(0) = 0$$

Thus $\omega_0 = \omega(0)$ must satisfy: $\int_0^{\omega_0} f(s)\,ds = 0.$

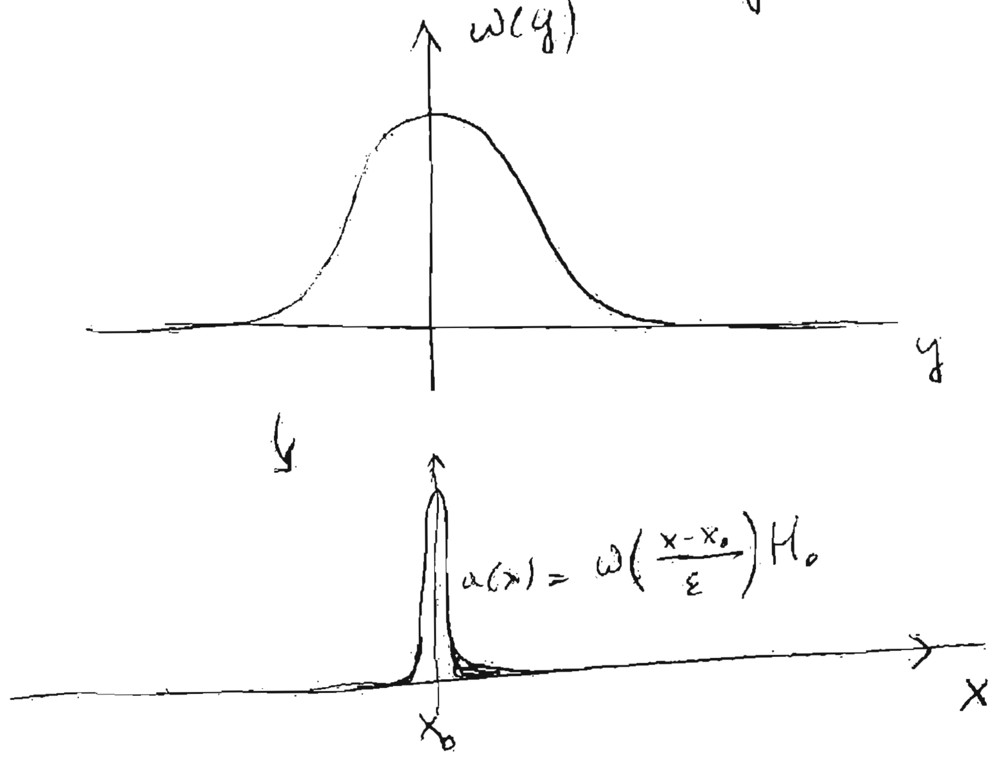




<u>Exercise</u>: $\boxed{\omega(y) = \frac{3}{2}\,\mathrm{sech}^2\left(\frac{y}{2}\right)}$

ω is a <u>homoclinic orbit</u>

with $\omega \to 0$ as $y \to \pm\infty$



So $a(x)$ looks like a <u>spike</u> near $x = x_0$

In the <u>outer region</u>, we have $a \to 0$ as $|x - x_0| \gg \varepsilon$

(since $\omega \to 0$ as $y \to \pm\infty$)

and we ignore $\varepsilon^2 a_{xx}$ $\Rightarrow$

$$-a + \frac{a^2}{h} = 0 \; ; \quad D h_{xx} - h + a^2 = 0$$

$$\Rightarrow a \equiv 0 \quad \text{and} \quad \begin{cases} D h_{xx} - h = 0 . \\ h'(-L) = 0 \end{cases}$$

By symmetry, we may assume $x_0 = 0$ for the steady state.

To determine H we integrate over the interface:

$$D\int_{x_0^-}^{x_0^+} h_{xx} - \int_{x_0^-}^{x_0^-} h + \int_{x_0^+}^{x_0^+} a^2 = 0$$

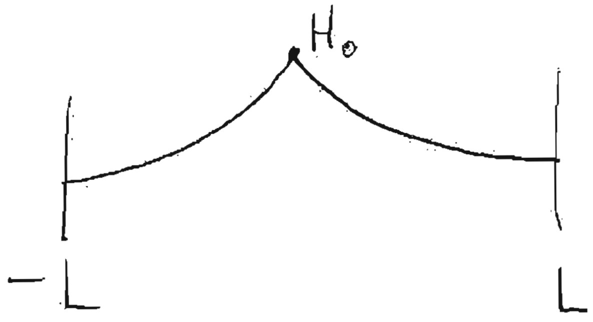
Now
$$\int_{x_0^-}^{x_0^+} a^2 \sim \int_{-\infty}^{\infty} A^2(y)\, dx$$
$$\sim \varepsilon \int_{-\infty}^{\infty} \omega^2(y)\, H_0^2\, dy$$
$$\sim H_0^2\, \varepsilon\, 6$$

$$\Rightarrow D\left(h'(x_0^+) - h'(x_0^-)\right) = -H_0^2\, \varepsilon\, 6$$

<u>So we have</u>:

$$(*) \begin{cases} D h'' - h = 0, & x \neq x_0 \\ h'(\pm L) = 0, & D h' \Big|_{x_0^-}^{x_0^+} = -H_0^2 6 \varepsilon \end{cases}$$

Note that h' is <u>discontinuous</u> at $x = x_0$;
[at least to leading order]; h looks like this:



An equivalent way to write $(*)$ is:

$$\begin{cases} D h'' - h = -H_0^2 \sigma \varepsilon \, \delta(x - x_0) \\ h'(\pm L) = 0 \end{cases}$$

I.E. $h(x) = \dfrac{\sigma \varepsilon H_0^2}{D} G(x, x_0)$ where

G is the Green's function satisfying

$$(G) \quad \begin{cases} G'' - \mu^2 G = -\delta(x - x_0) \\ G'(\pm L, x_0) = 0 \\ \mu = \sqrt{\frac{1}{D}} . \end{cases}$$

Next we solve (G): we write

$$G(x, x_0) = \begin{cases} A \dfrac{\cosh((x + L)\mu)}{\cosh((x_0 - L)\mu)}, & x < x_0 \\ A \cosh(\mu(x_0 + L)) \cosh(\mu(x - L)), & x > x_0 \end{cases}$$

[where we recall that $\cosh z = \dfrac{e^z + e^{-z}}{2}$]

By construction, $G'' - \mu^2 G = 0, \; x \neq x_0$

and $G(x, x_0)\big|_{x = x_0^-} = G(x, x_0)\big|_{x = x_0^+}$

and $G'(\pm L, x_0) = 0$.

To find A we integrate:

$$G'(x, x_0)\Big|_{x=x_0^-}^{x=x_0^+} = -1$$

$$\Rightarrow \quad A\mu \left\{ \sinh(\mu(x_0+L)) \cosh(x_0-L)\mu - \cosh((x_0+L)\mu) \sinh(\mu(x_0-L)) \right\} = 1$$

Now as we showed before, $G(x, x_0) = G(x_0, x)$ so that A is <u>independent</u> of x_0;

and we get: $\boxed{A = \dfrac{1}{2\mu \sinh(\mu L) \cosh(\mu L)}}$.

Now consider the case $\underline{x_0 = 0}$;

then we get $\quad G(0,0) = \dfrac{\cosh(\mu L)}{2\mu \sinh(\mu L)}$

and $H_0 \sim h(x_0) = G(0,0) \dfrac{6\varepsilon H_0^2}{D}$

$$\Rightarrow \quad H_0 = \frac{6\varepsilon H_0^2}{D} \frac{\coth(\mu L)}{2\mu}, \quad \mu = D^{-\frac{1}{2}}$$

$$\boxed{H_0 = \frac{D^{\frac{1}{2}}}{3\varepsilon} \tanh\left(D^{-\frac{1}{2}} L\right)}$$

Stability:

We linearize (GM) around the steady state

$$a(x,t) = a(x) + e^{\lambda t}\,\phi(x)$$
$$h(x,t) = h(x) + e^{\lambda t}\,\psi(x)\;;$$

$$\Rightarrow \quad (S)\;\begin{cases} \lambda\phi = \varepsilon^2\phi'' - \phi + 2\dfrac{a}{h}\phi - \dfrac{a^2}{h^2}\psi \\ \tau\lambda\psi = D\psi'' - \psi + 2a\phi \end{cases}$$

Consider the case $\underline{\tau = 0}$; and assume $\underline{x_0 = 0}$

In the <u>outer region</u>, we get

$$\begin{cases} D\psi'' - \psi = \delta(x)\left(-\int 2a\phi\right) \\ \psi'(\pm L) = 0 \end{cases}$$

$$\Rightarrow \quad \psi = G(x,0)\left(\frac{+2\int a\phi}{D}\right)$$

In the <u>inner region</u>, $x \sim x_0 = 0$, rescale:

$x = y\varepsilon$;

$\varphi(x) = \Phi(y)\, H_0$; $\quad h \sim H_0$

$a(x) = H_0\, \omega(y)$; $\quad \psi(x) \sim \psi_0 = \psi(0)$

$$\Rightarrow \begin{cases} \lambda \Phi = \Phi'' - \Phi + 2\omega \Phi - \dfrac{\omega^2}{H_0}\psi_0 \; ; \\[2ex] \psi_0 = G(0,0) \left(+ \dfrac{2 H_0 \varepsilon}{D} \displaystyle\int \omega(y)\, \Phi(y)\, dy \right) \end{cases}$$

where $G(0,0) = \dfrac{1}{2\mu} \coth(\mu L)$, $\mu = D^{-\frac{1}{2}}$;

$$H_0 = \frac{D^{\frac{1}{2}}}{3\varepsilon} \tanh(\mu L)$$

$$\Rightarrow \boxed{\frac{\psi_0 \ell}{H_0} = + \frac{1}{3} \int \omega(y)\, \Phi(y)}$$

$$\Rightarrow \boxed{\begin{aligned} &\lambda \Phi = L_0 \Phi - \frac{1}{3} \int \omega(y)\, \Phi(y)\; \omega^2 \\ &\text{where } L_0 \Phi = \Phi'' - \Phi + 2\omega \Phi \end{aligned}} \quad \text{(NLEP)}$$

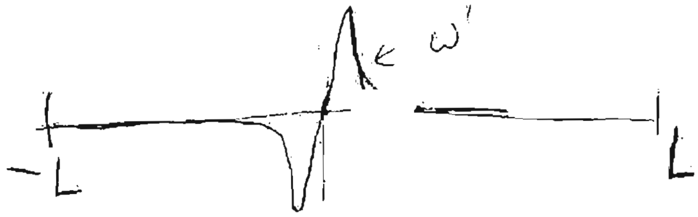
$$\boxed{\Phi'(y) \to 0 \text{ as } y \to \pm\infty .}$$

Note that $L_0 w_y = {}^{\prime\prime\prime} - w' + 2ww' = 0$

(since $L_0 w' = (w'' - w + w^2)' = 0$)

Moreover, $\int_{-\infty}^{\infty} w w' = 0.$

Thus $\lambda = 0$, $\Phi = w'$ is an eigenval/efunc pair to (NLEP). This implies that the original problem (S) has a <u>small eigenvalue</u>, the corresponding eigenfunction is $\varphi \sim w'(\frac{x}{\varepsilon})$ (to leading order)



Note that w' <u>has one root on $[-L, L]$</u>.

<u>Question</u>: Does there exist a positive $O(1)$ eigenvalue??

<u>Answer</u>: [Wei, "On single interior spike solutions of GM system: Uniqueness and spectral ~~est~~ estimates, 1997]

<u>Theorem</u>: Let $w = \frac{3}{2}\,\mathrm{sech}^2\left(\frac{y}{2}\right)$ be the ~~homoclinic~~ homoclinic solution to $\begin{cases} w'' - w + w^2 = 0 \\ w \to 0 \text{ as } y \to \pm\infty. \end{cases}$

Consider the eigenvalue problem

$$\text{(NLEP)} \quad \begin{cases} \lambda \varphi = L_0 \varphi - \gamma \dfrac{\int w \varphi}{\int w^2}\, w^2 \\[2ex] \varphi \to 0 \quad \text{as} \quad y \to \pm\infty \end{cases}$$

where $L_0 \varphi = \varphi'' - \varphi + 2 w \varphi$.

Then $\mathrm{Re}\,\lambda \leq 0$ iff $\gamma > 1$.

<u>Proof</u>: We will only consider the case where λ is real here.

<u>Remark</u>: When $\gamma = 0$, the problem $\lambda \varphi = L_0 \varphi$ is self-adjoint and from standard theory, the eigenvalues can be indexed

$$\lambda = \lambda_0 > \lambda_1 > \lambda_2 \cdots$$

with the corresponding eigenfunction

φ_k, $k=0,1,\dots$ having exactly k roots.

Since $\lambda=0$ corresponds to $\varphi=w'(y)$ having 1 root, $\lambda_1=0$ and $\Rightarrow$ $\boxed{\lambda_0>0}$. So (NLEP) is <u>unstable</u> <u>when $\gamma=0$.</u>

Note that $L_0 w = w^2$.

<u>Claim</u>: If $\gamma=1$ then $\lambda=0$, $\varphi=w$.

<u>Proof</u>: If $\varphi=w$ then

$$L_0\varphi - \frac{\int w\varphi}{\int w^2} w^2 = Lw - w^2 = 0.$$

Converse is also true: if $\lambda=0$ then either $\varphi=w'$ or $\varphi=w$ and $\gamma=1$.

<u>Proof</u>: We have

$$L_0\varphi = Aw^2, \quad \text{where } A=\frac{\gamma\int w\varphi}{\int w^2}.$$

So either $L_0\varphi=0$, $A=0$

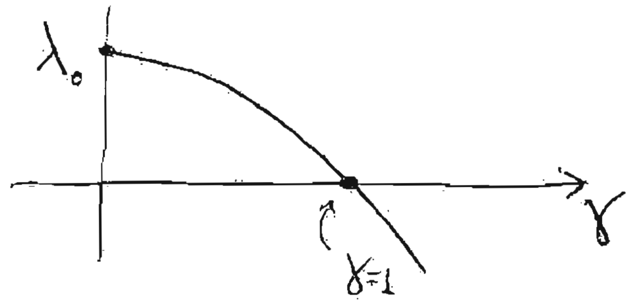
or $\varphi = Aw+\varphi_N$ where $\varphi_N\in\operatorname{Ker}L_0$

Moreover, $\underline{\mathrm{Ker}\, L_0 = \{ \omega' \}}$

So either $\varphi = \omega'$ or

$$\varphi = A\omega \Rightarrow A = \gamma \frac{\int \omega A \omega}{\int \omega^2} \Rightarrow \gamma = 1.$$

Thus, the eigenvalue $\lambda_0 > 0$ of $\lambda = L_0 \varphi$ crosses zero exactly when $\gamma = 1$.



Note that
$$\varphi \to L_0 \varphi - \gamma \frac{\int \omega \varphi}{\int \omega^2} \omega^2$$
is not self-adjoint. So in general, λ may be complex. See [Wei, 97] for more details.