## Mean first-passage time problems:

- A particle is undergoing a brownian motion inside domain $[0,1]$
- The left boundary is reflective whereas the right boundary is absorbing.

Q: How long does it take, on average, before the particle hits the right boundary? [i.e. "mean first passage time" MFPT]

Model: Let $u(x)$ denote the MFPT of a particle that is initially located at position "x".

Then $$\boxed{u(x) = \frac{1}{2}\left(u(x-\Delta x) + u(x+\Delta x)\right) + \Delta t}$$

Here, particle jumps a distance of $\Delta x$ in the time interval $\Delta t$, with equal probability either to the left or to the right.

Boundary conditions: $$\boxed{\begin{aligned} u'(0) &= 0 \\ u(1) &= 0 \end{aligned}} \quad \begin{aligned} &[\text{reflective}] \\ &[\text{absorbing}] \end{aligned}$$

Limit pblm:
$$u(x-\Delta x) = u - \Delta x u_x + \frac{(\Delta x)^2}{2} u_{xx} + O(\Delta x^3)$$
$$+ \quad + \quad + \quad +$$

$$u(x-\Delta x) + u(x+\Delta x) = 2u + (\Delta x)^2 u_{xx}$$

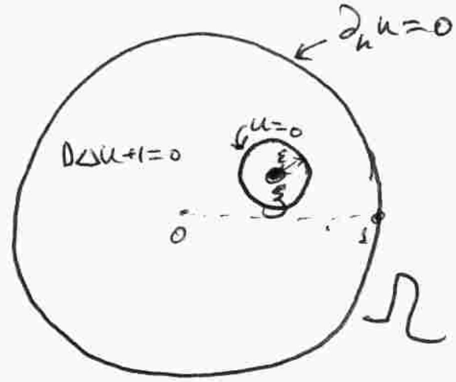
$$\Rightarrow \quad u(x) = u(x) + \frac{(\Delta x)^2}{2} u_{xx} + \Delta t$$

- The MFPT limit is:

$$\begin{cases} D\,\Delta u + 1 = 0 & \text{inside } \Omega \setminus B_\varepsilon(\xi) \\ \partial_n u = 0 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial B_\varepsilon(\xi) \end{cases}$$

where $\Omega = B_1(0)$ is the unit disk.

- WLOG, take $D=1$.




<u>Special case</u>: If $\xi = 0$ is at the origin of $\Omega$:

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + 1 = 0 \\ u(\varepsilon) = 0, \quad u'(1) = 0 \end{cases}$$

<u>Sol'n</u>: $$u_{rr} + \frac{1}{r} u_r = \frac{1}{r}(r u_r)_r = -1,$$

$$r u_r = -\frac{r^2}{2} + K$$

$$r=1: \; u_r(1) = -\frac{1}{2} + K = 0 \Rightarrow K = \frac{1}{2}$$

$$\Rightarrow u_r = -\frac{r}{2} + \frac{1}{2r}$$

$$u = -\frac{r^2}{4} + \frac{1}{2}\ln r + B,$$

$$0 = -\frac{\varepsilon^2}{4} + \frac{1}{2}\ln\varepsilon + B \Rightarrow B = -\frac{1}{2}\ln\varepsilon + \frac{\varepsilon^2}{4}$$

$$\Rightarrow \boxed{u(r) = \frac{1}{2}(\ln r - \ln\varepsilon) - \frac{r^2}{4} + \frac{\varepsilon^2}{4}}$$

Let $\tau = \frac{1}{|\Omega \setminus B_\varepsilon(\xi)|} \int_{\Omega \setminus B_\varepsilon} u(x)\,dx$ be the ~~ab~~ expected MFPT. (4

Here, we get:

$$\tau = \frac{\int_\varepsilon^1 u(r)\, 2\pi r\, dr}{\pi(1-\varepsilon^2)} \sim 2\int_0^1 u(r)\, r\, dr + O(\varepsilon)$$

$$\sim \int_0^1 \left( (\ln r - \ln \varepsilon) r - \frac{r^3}{2} \right) dr + O(\varepsilon)$$

$$\left\{ \int_0^1 \ln r\, r\, dr = -\frac{1}{4} \right.$$

$$\boxed{\tau = \frac{1}{2} \ln \frac{1}{\varepsilon} - \frac{3}{8} + O(\varepsilon)}$$

- Note that $\tau \sim \frac{1}{2} \ln \frac{1}{\varepsilon}$ as $\varepsilon \to 0$ and it grows ~~to~~ $\tau \to \infty$ as $\varepsilon \to 0$ but very slowly.

# General problem:

Consider the Neumann Green's Function:

$$\begin{cases} \Delta G + 1 = A\,\delta(x-\xi), & x,\xi \in \Omega \\ \partial_n G = 0, & x \in \partial\Omega,\ \xi \in \Omega \\ \int_\Omega G(x,\xi)\,dx = 0 \end{cases}$$

- The constant "A" is determined via a solvability condition:

  - $\int_\Omega (\Delta G + 1)\,dx = \int_\Omega A\,\delta(x-\xi) = A$
  - $\int_\Omega \Delta G\,dx = \int_{\partial\Omega} \underbrace{\partial_n G}_{0}\,dS(x) = 0$
  - $\int_\Omega 1\,dx = \text{area}(\Omega) =: |\Omega|$

$$\Rightarrow A = |\Omega| \qquad \boxed{\Delta G + 1 = |\Omega|\,\delta(x-\xi), \quad x,\xi \in \Omega}$$

- If $|\Omega| = B_1(0)$ is a unit disk, then $G$ is known explicitly:

$$G(x,\xi) = \frac{1}{2}\left[\ln|x-\xi| + \ln\left|x|\xi| - \frac{\xi}{|\xi|}\right| - \frac{1}{2}|\xi|^2 - \frac{1}{2}|x|^2 + \frac{3}{4}\right]$$

- More generally, note that the leading-order behaviour of $G(x, \xi)$ as $x \to \xi$ is:

$$G(x, \xi) \sim \frac{|\Omega|}{2\pi} \ln |x - \xi| \quad \text{as } x \to \xi$$

- This is because the sol'n to $\Delta u(x) = \delta(x)$ on all of $\mathbb{R}^2$ is given by $u(x) = \frac{1}{2\pi} \ln |x|$

- So we define the regular part of $G$ to be what is left after "peeling off" the log singularity:

Let $$\boxed{R(x, \xi) = G(x, \xi) - \frac{|\Omega|}{2\pi} \ln |x - \xi|}$$

- Back to $u(x)$: Take $u(x) = G(x, \xi) + v(x)$,

then $\Delta u + 1 = |\Omega| \delta(x - \xi) + \Delta v \stackrel{!}{=} 0$

so that $\Delta v = 0$ for $x \in \Omega \setminus B_\varepsilon(\xi)$,

and moreover, $\partial_n v = 0$ for $x \in \partial\Omega$.

To determine $v$, let's rescale $u$ in the "inner variabl

near $x \sim \xi$: i.e. change var:

$$\boxed{x = \xi + \varepsilon y}, \quad u(x) = \mathcal{U}(y).$$

Then the problem for $\mathcal{U}$ becomes:

$$\frac{\Delta \mathcal{U}}{\varepsilon^2} + 1 = 0$$

so to leading order, we have:

$$\begin{cases} \Delta \mathcal{U} = 0, & |y| > 1 \quad [\text{outside trap}] \\ \mathcal{U} = 0, & |y| = 1 \quad [\text{on } \partial B_\varepsilon(\xi)] \end{cases}$$

The sol'n is given by:

$$\mathcal{U}(y) = A \ln |y| \quad \text{where}$$

A is to be found by matching to the outer region:

- Expand $u(x)$ near $x = \xi + \varepsilon y$: we have:

$$u(x) = \frac{|\Omega|}{2\pi} \ln |x - \xi| + R(x, \xi) + V(x)$$

$$= \frac{|\Omega|}{2\pi} \ln |\varepsilon y| + R(\xi + \varepsilon y, \xi) + V(\xi + \varepsilon y)$$

$$= \frac{|\Omega|}{2\pi} \ln |y| + \frac{|\Omega|}{2\pi} \ln \varepsilon + R(\xi, \xi) + V(\xi) + O(\varepsilon)$$

$$\sim A \ln |y| \implies \boxed{A = \frac{|\Omega|}{2\pi}}; \text{ and}$$

$$\boxed{\frac{|\Omega|}{2\pi} \ln \varepsilon + R(\xi, \xi) + V(\xi) = 0}$$

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$\cdots$ So $V(\xi)$ satisfies: $\begin{cases} \Delta V = 0, & x \neq \xi \\ \partial_n V = 0, & x \in \partial\Omega \\ V(\xi) = \frac{|\Omega|}{2\pi} \ln\left(\frac{1}{\varepsilon}\right) - R(\xi,\xi) \end{cases}$

Then $v(x) = V(\xi)$, and

$$\tau \sim \frac{1}{|\Omega|}\int u = \frac{1}{|\Omega|}\underbrace{\int G\,dx}_{0} + V(\xi) = \frac{|\Omega|}{2\pi}\ln\left(\frac{1}{\varepsilon}\right) - R(\xi,\xi)$$

- If $\Omega = B_1$ we get: $R(\xi,\xi) = \frac{1}{2}\ln\left(1-|\xi|^2\right) - \frac{1}{2}|\xi|^2 + \frac{3}{8}$

$$\boxed{\tau \sim \frac{1}{2}\ln\frac{1}{\varepsilon} - \frac{1}{2}\ln\left(1-|\xi|^2\right) + \frac{1}{2}|\xi|^2 - \frac{3}{8}}$$

- Remark: if $\xi = 0 \Rightarrow \tau \sim \frac{1}{2}\ln\frac{1}{\varepsilon} - \frac{3}{8}$, in agreement with "direct" calculation done previously

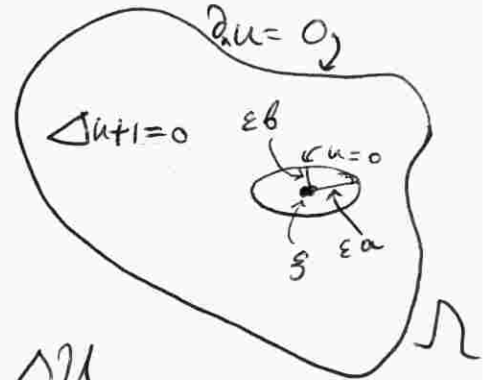
- $$\frac{\partial\tau}{\partial|\xi|} = \left(\underbrace{\frac{1}{1-|\xi|^2} + 1}\right)|\xi|$$

$$= |\xi|\left(\frac{2-|\xi|^2}{1-|\xi|^2}\right)$$

$\Rightarrow \tau$ has a min at $\xi = 0$

$\Rightarrow$ It is best to place the trap at the center to minimize expected MFPT.

<u>Elliptical trap</u>: Suppose the trap has the shape of an ellipse centered at $\xi$ whose axes have radii $\varepsilon a$, $\varepsilon b$:




- <u>Inner problem</u>:

Let $x = \xi + \varepsilon y$,

$u(x) = U(y)$. Then $\dfrac{\Delta U}{\varepsilon^2} + 1 = 0$

So to leading order, we get:

$$\boxed{\begin{cases} \Delta U = 0 & \text{for } y \ \underline{\text{outside}} \ \mathcal{E}_0 \\ U = 0 & \text{for } y \ \underline{\text{on}} \ \mathcal{E}_0 \end{cases}}$$

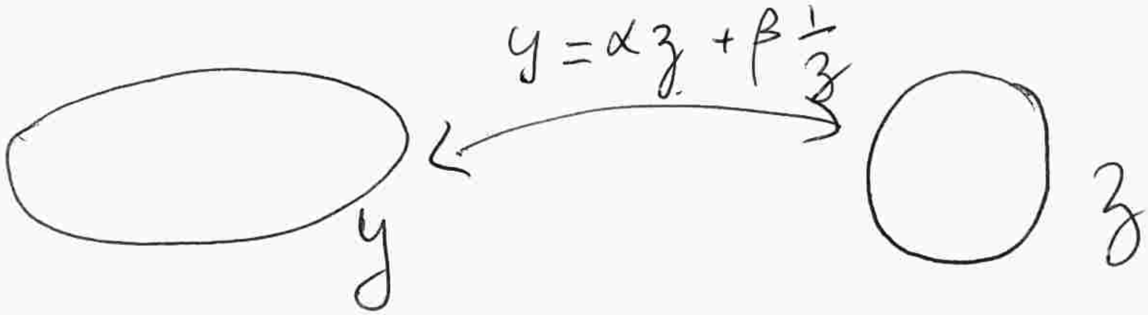

Where $\mathcal{E}_0$ is an ellipse $\partial \mathcal{E}_0 = \{(a\cos\theta, b\sin\theta), \ \theta = 0 \ldots 2\pi\}$

Moreover, $\boxed{U \sim \dfrac{|\Omega|}{2\pi} \ln|y| \quad \text{as } y \to \infty}$

To determine $U$, use analytic mapping to map an ~~ellipse~~ outside of ellipse into the outside of a unit disk:

$$y = \alpha z + \beta \frac{1}{z}, \quad \text{where}$$

$\alpha, \beta$ are to be determined [in terms of $a, b$]




For $z \in \partial B_1(0)$, write $z = e^{i\theta}$, then

$$y = \big((\alpha+\beta)\cos\theta,\ (\alpha-\beta)\sin\theta\big)$$

This shows that the map ~~transforms~~ takes $\partial B_1$ to $\partial \mathcal{E}_0$, provided that

$$\alpha+\beta = a, \quad \alpha - \beta = b \implies \boxed{\alpha = \frac{a+b}{2},\ \beta = \frac{a-b}{2}}$$

Since analytic functions preserve harmonicity, we have $\mathcal{U} \sim \frac{|\Omega|}{2\pi} \ln |z|$ satisfies

$\Delta \mathcal{U} = 0$.

We wish to know the behaviour of $\mathcal{U}$ for <u>large</u> $y$. Then we have $z \sim \frac{1}{\alpha} y$, $|y| \gg 1$, so that:

$$\mathcal{U} \sim \frac{|\Omega|}{2\pi} \ln \left| \frac{y}{\alpha} \right|$$

$$\boxed{\mathcal{U} \sim \frac{|\Omega|}{2\pi} \left\{ \ln|y| + \ln \frac{2}{a+b} \right\} + O\left(\frac{1}{y}\right)}$$

From before, recall that

$$u(x) \sim \frac{|\Omega|}{2\pi} \ln|y| + \frac{|\Omega| \ln \varepsilon}{2\pi} + R(\xi,\xi) + V_0 + \cdots$$

$$\sim \frac{|\Omega|}{2\pi} \left\{ \ln|y| + \ln \frac{2}{a+b} \right\} + \cdots$$

$$\Rightarrow \boxed{V_0 = \frac{|\Omega|}{2\pi} \ln \frac{1}{\varepsilon} - R(\xi,\xi) + \ln \frac{2}{a+b}}$$

"Log capacitance" of ellipse

- The last term shows the effect of the shape on MFPT $\tau \sim V_0$.