Rayleigh oscillator, multiple scales analysis

$$u'' + u = \varepsilon\left(1 - \frac{u'^2}{3}\right)u', \quad u(0)=1,\ u'(0)=0.$$

Expand:

$$u = u_0 + \varepsilon u_1, \quad u_0 = u_0(t,\tau), \quad \tau = \varepsilon t$$

$$\Rightarrow \begin{cases} u_0'' + u_0 = 0 \\ u_1'' + u_1 = u_0'\left(1 - \frac{u_0'^2}{3}\right) - 2u_{0t\tau} \end{cases}$$

$$u_0 = A(\tau)\cos(t + \varphi(\tau)) \qquad u_{0t\tau} = -A_\tau \sin\theta + A\varphi_\tau \cos\theta$$

$$u' = -A\sin\theta, \quad \theta = t + \varphi$$

$$u'^3 = -A^3\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)$$

$$\Rightarrow u_1'' + u_1 = \sin\theta\left[-A + \frac{A^3}{4} + 2A_\tau\right] + \cos\theta\left[2A\varphi_\tau\right]$$

+ non-secular

$$\Rightarrow 2A_\tau = A - \frac{A^3}{4}, \quad \varphi_\tau = 0, \quad A(0)=1, \quad \varphi(0)=0$$

$$\Rightarrow \begin{cases} A = \dfrac{2}{\sqrt{1+3e^{-\tau}}} \to 2 \text{ as } \tau \to \infty \\ u = A(\varepsilon t)\cos(t) \end{cases}$$

```
> restart;
> ode := diff(u(t),t,t)+u(t)=eps*(1-diff(u(t),t)^2/3)*diff(u(t),t);
```

$$ode := \left(\frac{d^2}{dt^2}\mathrm{u}(t)\right) + \mathrm{u}(t) = eps\left(1 - \frac{1}{3}\left(\frac{d}{dt}\mathrm{u}(t)\right)^2\right)\left(\frac{d}{dt}\mathrm{u}(t)\right)$$

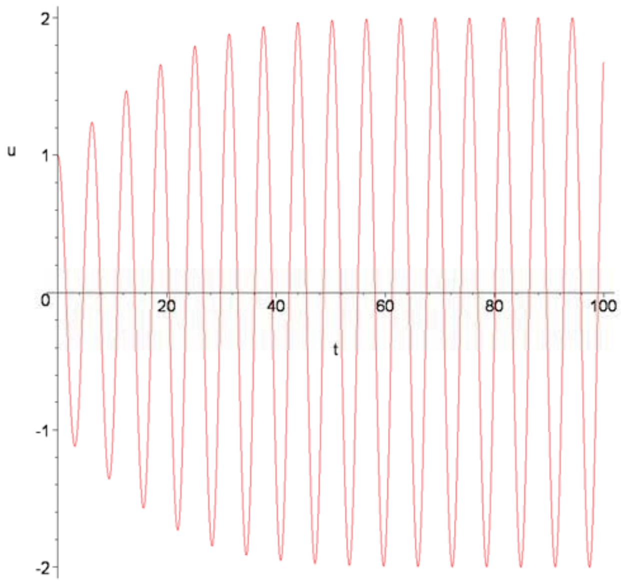
```
> eps := 0.1;
```

$$eps := 0.1$$

```
> sol := dsolve({ode, u(0)=1, D(u)(0)=0}, numeric, maxfun=0);
```

$$sol := \mathbf{proc}(x\_rkf45) \;\ldots\; \mathbf{end\ proc}$$

```
>
> with(plots):
  odeplot(sol, [t, u(t)], 0..100, numpoints=10000);
```



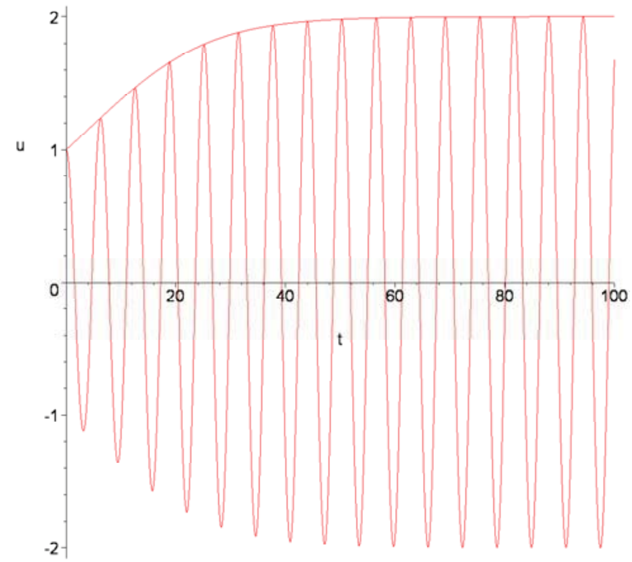

```
> pic := %:
> A := 2/sqrt(1+3*exp(-tau));
```

$$A := \frac{2}{\sqrt{1 + 3\,\mathbf{e}^{(-0.1\,t)}}}$$

```
> tau := eps*t;
```

$$\tau := 0.1\,t$$

```
> with(plots):
> display(pic, plot(A, t=0..100));
```

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>
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