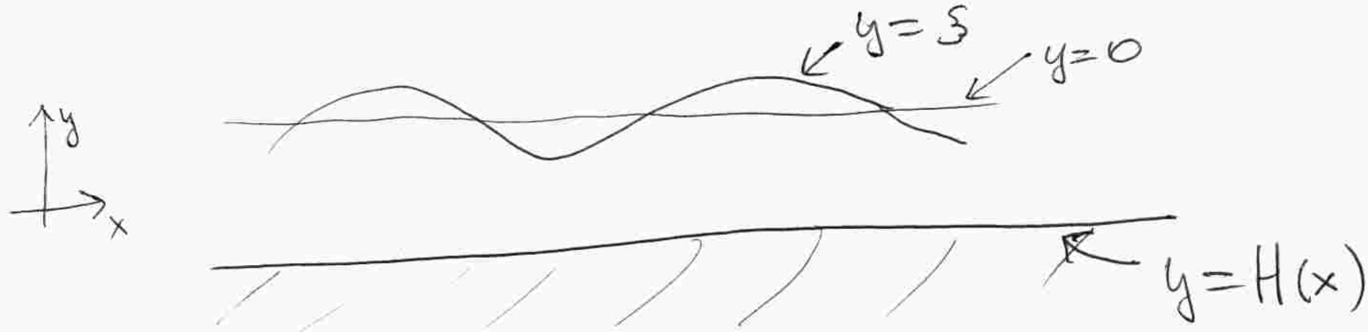
Shallow Water Waves

Ref: • Frederic Y.M. Wan, Mathematical models and their applications, 1989

Consider a fluid (water) inside a channel (e.g. river). Assume that fluid velocity only depends on the positional length "x" within the channel but <u>not</u> on the depth "y". Also ignore friction at the sides, viscosity effects etc. Under these assumptions, the fluid motion can be effectively described by a one-dimensional PDE system which we now derive.

• Suppose the surface is given by $y = \xi = \xi(x)$ and the bottom is at $y = -H$



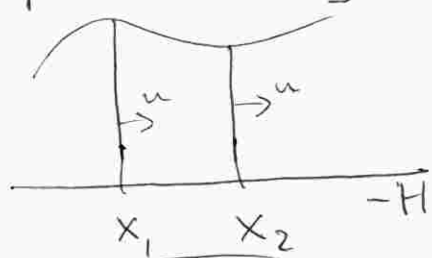
Let $\rho \equiv$ density $\quad \rho = \rho(\cancel{x, y, t})$

$P \equiv$ pressure $\quad P = P(x, y, t)$

$u \equiv$ water velocity $\quad u = u(x, \cancel{y}, t)$

<u>Assume</u> ρ is <u>constant</u> [incompressible fluid]

<u>Conservation of mass:</u>



$$M(t) = \int_{x_1}^{x_2} \left(\int_{-H}^{\xi(x)} \rho \, dy \right) dx$$

$$\boxed{h = \xi + H}$$

$$= \int_{x_1}^{x_2} \rho(x) h(x) \, dx$$

$$M(t+\Delta t) - M(t) \cong \left[\int_{-H}^{\xi(x_1)} u \rho \, dy - \int_{-H}^{\xi(x_2)} u \rho \, dy \right] \Delta t$$

$$= -(u \rho h) \Big|_{x_1}^{x_2}$$

$$\Rightarrow \quad \cong \int_{x_1}^{x_2} (\rho h)_t \, \Delta t \, dx$$

$$\Rightarrow \boxed{\int_{x_1}^{x_2} (\rho h)_t + (u \rho h) \Big|_{x_1}^{x_2} = 0}$$

<u>Continuum limit $x_1 \to x_2$</u>: If ρ, u, h are cont. and diff.

then $(\rho h)_t + (u \rho h)_x = 0 \Rightarrow \boxed{h_t + (uh)_x = 0}$

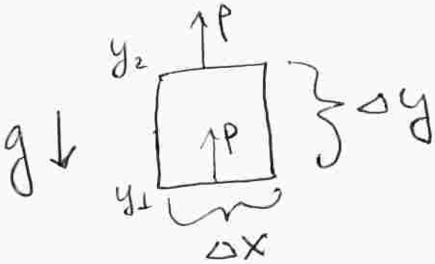
"Conservation of mass"

<u>Hydrostatic equilibrium</u>: Assume no motion of water in the vertical direction

- Given a piece of water, compute the force acting in horizontal direction:

$$F \equiv \text{gravity} + \text{pressure} = 0$$

· gravity $\equiv -g\,\Delta x\,\Delta y\,\rho$



· pressure $\equiv \Delta x\, p(y_1) + \Delta x\, p(y_2)$

$$\sim \Delta x\,\Delta y\, p_y$$

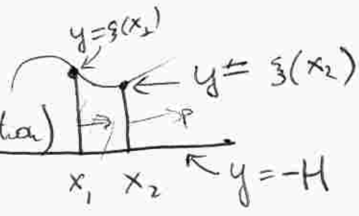
$$\Rightarrow \boxed{p_y = -g\,\rho}$$

$$\Rightarrow \boxed{p = -g\rho(y-\xi)}$$ where the atmospheric pressure is normalized to zero.

<u>Momentum balance in horizontal direction:</u>

Newton's law ("f=ma"): $\frac{d}{dt}(mu) = F$, where $u \equiv$ velocity, $m \equiv$ mass

Consider a "slice" of fluid:



- its momentum (in horizontal direction)

$$\mu(t) = \int_{x_1(t)}^{x_2(t)} \rho\, u\, h(x)\,dx, \qquad h = \xi + H$$

Then $$\frac{d\mu}{dt} = \int_{x_1}^{x_2} (\rho u h)_t + \rho u h\Big|_{x_2} \underbrace{\frac{\partial}{\partial t} x_2(t)}_{u(x_2)} - \rho u h\Big|_{x_1} \underbrace{\frac{\partial}{\partial t} x_1(t)}_{u(x_1)},$$

$$= \int_{x_1}^{x_2} (\rho u h)_t + (\rho u^2 h)\Big|_{x_1}^{x_2}$$

- The only force acting in the horizontal direction is the pressure at the boundaries:

$$F_{Horiz} \equiv \int_{-H}^{\xi(x_1)} P(x_1, y)\, dy - \int_{-H}^{\xi(x_2)} P(x_2, y)\, dy$$

Now $$\int_{-H}^{\xi} P\, dy = -\frac{g\rho}{2}(y-\xi)^2\Big|_{-H}^{\xi} \quad \cancel{+P_0 h(x)}$$

$$= +\frac{g\rho}{2} h^2$$

$$\Rightarrow F_{horiz} = -\frac{g\rho}{2} h^2 \Big|_{x_1}^{x_2} = \frac{d}{dt}\mu$$

$$\Rightarrow \boxed{\int_{x_1}^{x_2} (\rho u h)_t + \left(\rho u^2 h + \frac{\rho g}{2} h^2\right)\Big|_{x_1}^{x_2} = 0}$$

"Conservation of momentum"

Continuum version:

$$\begin{cases} (\rho h)_t + (u\rho h)_x = 0 \\ (\rho u h)_t + \left(\rho u^2 h + \dfrac{\rho g h^2}{2}\right)_x = 0 \end{cases}$$

$$\Rightarrow h_t = -u h_x - u_x h$$

$$u_t h + \underbrace{u h_t}_{-u^2 h_x - u u_x h} + 2u u_x h + u^2 h_x + g h h_x = 0$$

$$\Rightarrow \quad (*) \quad \boxed{\begin{cases} h_t + (uh)_x = 0 \\ u_t + u u_x + g h_x = 0 \end{cases}}$$

Note that (*) assumes that u, h are <u>smooth</u> which in practice means the absence of <u>shocks</u>. We will aslo consider a propagating-shock solution [a "bore"]; in this case, we will use the integral formulation:

$$\begin{cases} \dfrac{d}{dt}\left(\displaystyle\int_{x_1}^{x_2} \rho h\right) + (\rho u h)\Big|_{x_1}^{x_2} = 0 & \forall x_1, x_2 \\ \dfrac{d}{dt}\left(\displaystyle\int_{x_1}^{x_2} \rho u h\right) + \left(\rho u^2 h + \dfrac{\rho g h^2}{2}\right)\Big|_{x_1}^{x_2} & \forall x_1, x_2 \end{cases}$$

Small perturbations: Suppose that $\boxed{\xi \ll H, \text{ and } u \ll 1}$

Recall, $h = H + \xi \sim H$; $h_x \sim \xi_x$

$$\Rightarrow \begin{cases} \xi_t + H u_x \sim 0 \\ u_t + g \xi_x = 0 \end{cases}$$

$$\Rightarrow \quad \xi_{tt} = -H u_{xt} = gH \xi_{xx}$$

$\Rightarrow$ get a wave eq'n:

$$\boxed{\xi_{tt} = c^2 \xi_{xx}, \quad c = \sqrt{gH} \text{ is the wave speed}}$$

and $\boxed{u_{tt} = c^2 u_{xx}}$

Given i.c. $\begin{cases} u(x,0) = u_0(x) \\ u_x(x,0) = V_0(x), \end{cases}$

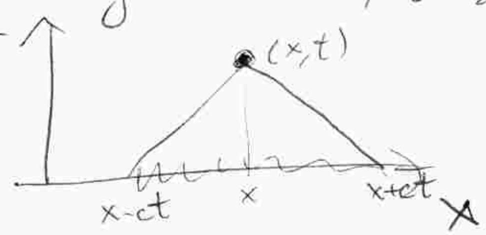
the sol'n is given by

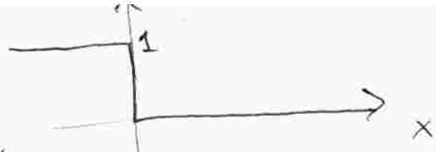
$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V_0(s)\,ds$$

[D'Alembert formula]

Cone of dependence: "c" is the speed of propagation of disturbances;

Sol'n at (x,t) depends only on u_0, V_0 on in $[x-ct, x+ct]$

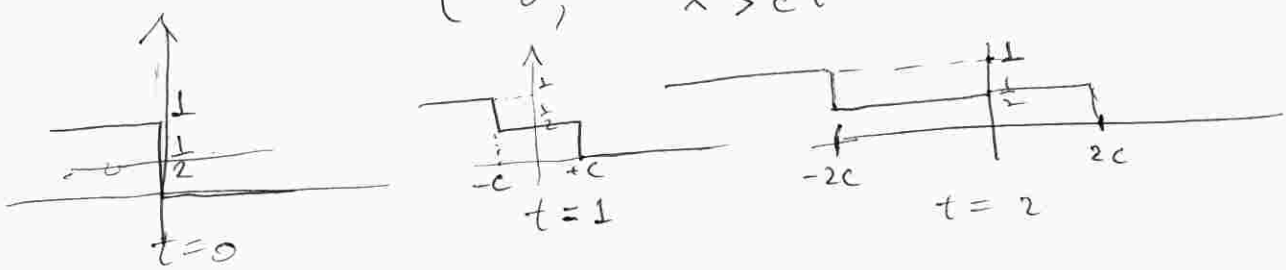


Eg If $u_0(x) =$  $; \; v_0 = 0$

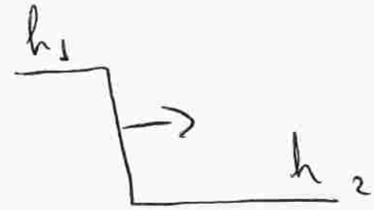
$\begin{cases} 1, & x<0 \\ 0, & x>0 \end{cases}$

then $u(x,t) = u_0(x+ct) + u_0(x-ct)$

$$= \begin{cases} 1, & x < -ct \\ \frac{1}{2}, & x \in (-ct, ct) \\ 0, & x > ct \end{cases}$$



<u>Bore</u>: A solution with



$$\begin{cases} h = \begin{cases} h_1, & x < x_s \\ h_2, & x > x_s \end{cases} \\ u = \begin{cases} u_1, & x < x_s \\ u_2, & x > x_s \end{cases} \end{cases}$$

Where $x_s = x_s(t)$ is the location of the bore which moves to the right in time [we assume that $h_1 > h_2$ and $u_1, u_2 \geq 0$]

<u>Q</u>: • What is the velocity $\dot{x}_s(t)$ of the bore?
• What is the relationship between h_1, h_2, u_1, u_2?

• Recall the integral formulation:

$$\frac{\partial}{\partial t}\left(\int_{x_1}^{x_2} h\right) + uh\Big|_{x_1}^{x_2} = 0$$

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} uh + \left(u^2 h + \tfrac{1}{2} g h^2\right)\Big|_{x_1}^{x_2} = 0$$

• Take $x_1 < x_s < x_2$. Then:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} h = \frac{\partial}{\partial t}\left[\int_{x_1}^{x_s(t)} h + \int_{x_s(t)}^{x_2} h\right] = \dot{x}_s h_1 - \dot{x}_s h_2$$

$$\Rightarrow \begin{cases} \dot{x}_s(h_1 - h_2) + u_2 h_2 - u_1 h_1 = 0 \\ \dot{x}_s(u_1 h_1 - u_2 h_2) + u_2^2 h_2 + \tfrac{1}{2} g h_2^2 - u_1^2 h_1 - \tfrac{1}{2} g h_1^2 = 0 \end{cases}$$

$\Rightarrow$ <u>Two</u> algebraic constraints for <u>five</u> variables $\dot{x}_s, h_1, u_1, h_2, u_2$.

Ex: If $u_2 = 0$ then we get:

$$\boxed{\dot{x}_s = \sqrt{g}\,\frac{\sqrt{(h_1+h_2)/2}}{\sqrt{h_2+1-h_1}}, \quad u_1 = \dot{x}_s\left(1-\frac{h_2}{h_1}\right)}$$

with $\boxed{h_1 - h_2 < 1}$ [else no bore]

- since $h_1 > h_2$, we see that $u_1 < \dot{x}_s$.

Let $c = \sqrt{gh}$;

$$\begin{cases} 2(c_t + u c_x) + c\, u_x = 0 \\ u_t + u u_x + 2 c c_x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (u+2c)_t + (u+c)(u_x + 2c_x) = 0 \\ (u-2c)_t + (u-c)(u_x - 2c_x) = 0 \end{cases}$$

Let $\xi(x,t)$ be solution to $\quad \xi_t + (u+c)\xi_x = 0$

$\eta(x,t)$ " $\quad \eta_t + (u-c)\eta_x = 0$

Note: If $F = f(\xi(x,t))$ then F also satisfies:

$$F_t + (u+c) F_x \quad (*)$$

The converse is also true: $F = F(\xi) \Leftrightarrow F$ solves $(*)$

Thus: $\boxed{\begin{cases} u + 2c = f(\xi) \\ u - 2c = g(\eta) \end{cases}}$ for some functions f, g

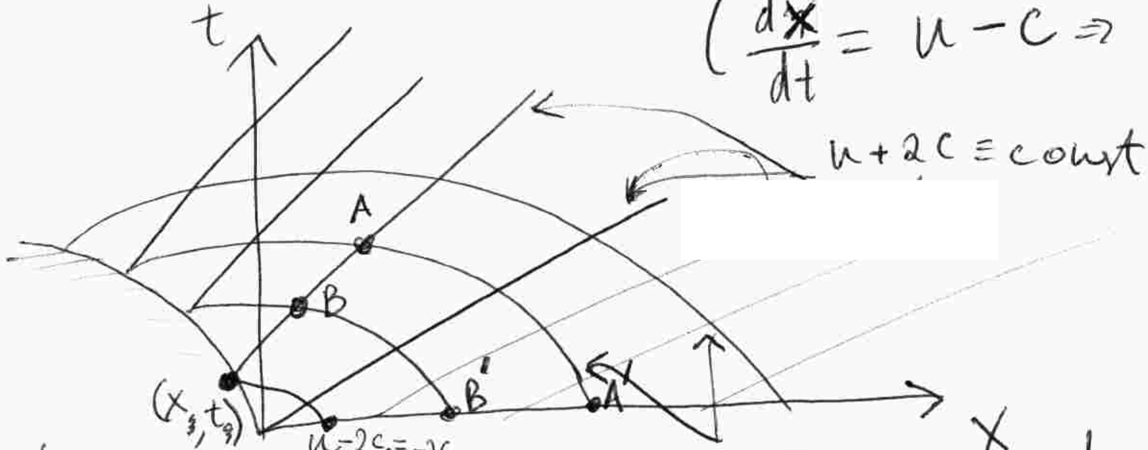
Piston plm: a piston moves to the left; its position is given by $x_\omega = -\beta t^2$; assume $c = c_0$ at $t=0$, $x > 0$

x_ω

I.C.: $\begin{cases} \begin{cases} c = c_0 \\ u = 0 \end{cases} \text{ at } t = 0 \text{ and } x \geq 0 \\ \& \; u = -2\beta t \text{ along } x = -\beta t^2 \end{cases}$

Characteristic coords:

$$\begin{cases} \dfrac{dx}{dt} = u + c \Rightarrow u + 2c \equiv \text{const} \\[2ex] \dfrac{dx}{dt} = u - c \Rightarrow u - 2c \equiv \text{const} \end{cases}$$



Given pts A, B along $\xi = \xi_0$ const.,

We have: $u_A + 2c_A = u_B + 2c_A$

Let A' be the intersection of characteristic $\eta \equiv \text{const}$ which also contains A, with the x-axis. Similar for B'. Then $u_A - 2c_A = u_{A'} - 2c_{A'}$

$$= -2c_0$$

and similar $u_B - 2c_B = -2c_0$

$$\Rightarrow \boxed{u_A = u_B \text{ and } c_A = c_A}$$

But then $\dfrac{dx}{dt} = u_A + c_A = u_B + c_B \equiv \text{const}$

$\Rightarrow$ characteristic curves for ξ are straight lines

$$\boxed{x = x_\xi + (u_\xi + c_\xi)(t - t_\xi)}$$

Choose (x_ξ, t_ξ) to lie on the initial data

$$t_\xi = X_\omega(t_\xi) = \qquad -\beta t^2 .$$ Then

$$u_\xi = -2\beta t_\xi \,; \quad c_\xi \text{ is not specified from i.c.}$$

But then $u - 2c$ is const. along $\frac{dx}{dt} = u - c$;

so we get
$$\begin{cases} u_\xi - 2c_\xi = -2c_0 \\ u_\xi = -2\beta t_\xi \end{cases}$$

Then $c_\xi = \frac{u_\xi}{2} + c_0 = -\beta t_\xi + c_0$

$$\Rightarrow \boxed{x = -\beta t_\xi^2 + (c_0 - 3\beta t_\xi)(t - t_\xi)} \quad (\star)$$

Note: Choosing $t_\xi = 0$, $x_\xi > 0$, then $u_\xi = 0$, $c_\xi = c_0$

$$\Rightarrow x = x_\xi + c_0 t$$

with $u = u_\xi = 0$, $c = c_0$.

Thus: $\boxed{\begin{matrix} c = c_0 \text{ whenever } x \geq c_0 t \\ u = 0 \end{matrix}}$

[this is consistent with the fact that c is the "speed of propagation"]

That is, the ~~fl~~ fluid is at rest to the right of $x \geq c_0 t$.

Conclusion:

If $0 \leq x \leq t c_0$ then

$$\begin{cases} c = c_\xi = c_0 - \beta t_\xi \\ u = u_\xi = -2\beta t_\xi \end{cases}$$

where $t_\xi = t_\xi(x,t)$ is given implicitly by ($\star$).

If $t c_0 \leq x$ then $c = c_0$, $u = 0$

[still water]