Model of insect swarming

- Swarms are often observed in insect formations, flocks of birds, schools of fish and bacterial colonies.
- They tend to maintain their shape over long time periods.
- What is the minimal mathematical model that exhibits ~~&~~ such behaviour?
- Consider a particle model where each insect is represented by a particle whose position is given by x_j, $j = 1 \dots N$.
- Kinematic model ignores the acceleration effects [first order]. The simplest

such model is:

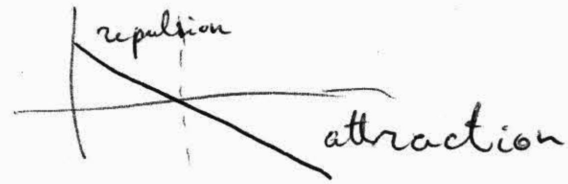
$$\frac{d}{dt} x_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} F(|x_i - x_j|) \frac{x_j - x_i}{|x_j - x_i|} \tag{1}$$

Here, $F(|x_i - x_j|)$ is the force exherted by x_i on x_j & it is assumed that all particles are identical.

- If $F(|x_i - x_j|) > 0$ then x_i and x_j repulse each oth[er]
  If $F(|x_i - x_j|) < 0$ then x_i and x_j attract each-other

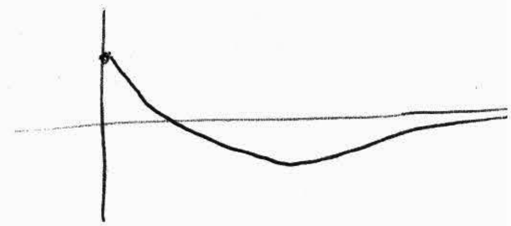
- We consider an interaction force $F(r)$ which is attractive at large distances and is repulsive at short distances, such as for example:

(2) $$F(r) = 1 - r$$



- Often, a Morse force is used, it is:

(3) $$F(r) = e^{-r} - G e^{-r/L}$$



with $G < 1$ and $L > 1$

- Q: Does (2) or (3) leads to swarming?
- Numerically, distinct regimes are observed
  - If $F(r) > 0$ for all $r > 0$, the particles collapse to a single point, $X_j(t) \to \overline{X}$ as $t \to \infty$ $\forall j$
  - If $F(r) < 0$ for all $r < 0$, the particles spread, $|X_j| \to \infty$ as $t \to \infty$ $\forall$ $j = 1 \dots N$
  - If $|X_j(t)| < R$

particles form a swarm.

Numerically, two types of swarming behaviour are observed: either the swarm size increases as N is increased, or else the swarm size approaches some limit R independent of N. The two types are illustrated in 2-D by taking Morse force:

$$F(r) = e^{-r} - G e^{-r/L}$$

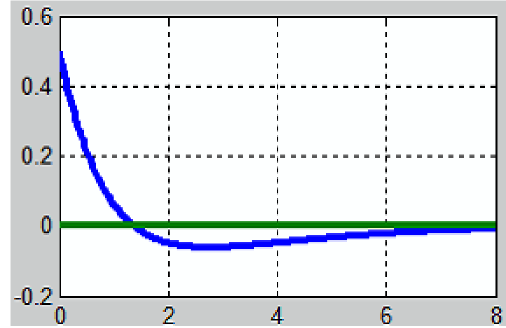
- ~~If~~ $G = 0.5$, $L = 2$, the swarm radius is $R \approx 1.5$ as $N \to \infty$ and appears to be indep. of N.

- If $G = 0.5$, $L = 1.2$, the swarm radius increases with N while the particle density ~~b~~remains the same. This is illustrated in the figure below.

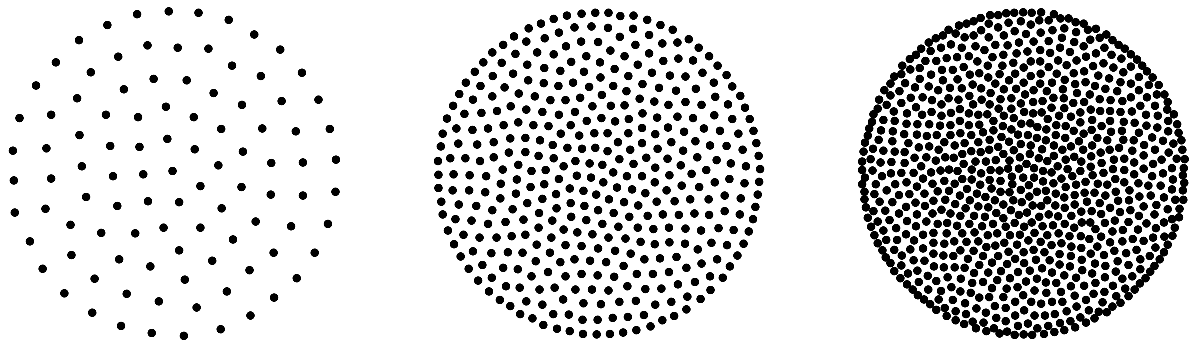
- The former case is called catastropic while the latter is called called H-stable.

Example of catastrophic regime:

$$F(r) = e^{-r} - 0.5e^{-r/2}$$



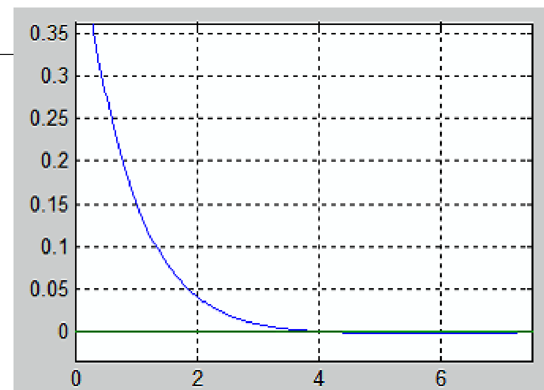
r=1.3978 r=1.43535 r=1.44716



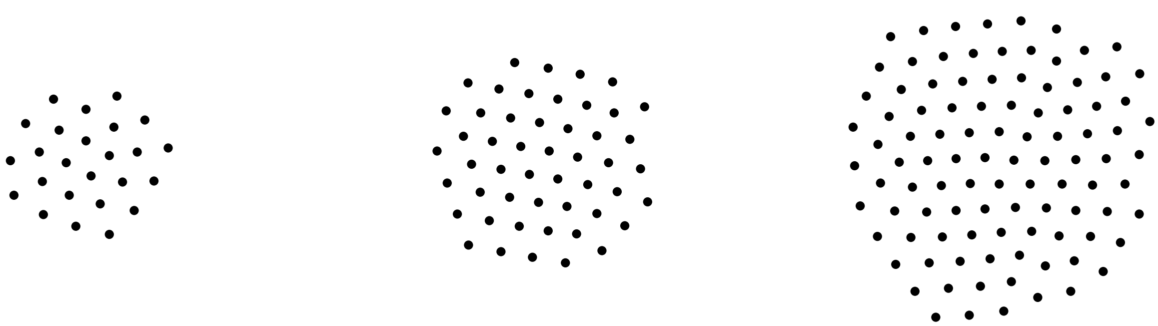
N=100 N=400 N=800

Example of h-stable regime:

$$F(r) = e^{-r} - 0.5e^{-r/1.2}$$



r=9.56367 r=13.3742 r=19.3298



N=25 N=50 N=100

Continuum limit $N \to \infty$:

Let $\rho(x,t) \equiv$ density of particles at position x and time t; that is,

$$\int_\Omega \rho(x,t)\,dx \equiv \frac{\#\text{ of particles inside } \Omega}{N}$$

We assume that $\rho(x,t)$ becomes smooth in the limit $N \to \infty$. In addition, define

$V(x,t) \equiv$ velocity field [i.e. vector].

Then $$V(x,t) = \int F(|x-y|)\frac{x-y}{|x-y|}\rho(y)\,dy.$$

Moreover, for $\Delta t \to 0$ we have:

$$\int_\Omega \rho(x,t+\Delta t) - \rho(x,t) \equiv \text{net density in time } \Delta t$$

$$= -\int_{\partial\Omega} \vec{V}\cdot\hat{n}\,\rho \quad dS(x)\Delta t \quad \left[\begin{array}{l}\text{amount of mass}\\ \text{crossing boundary in } \Delta t\end{array}\right]$$

$$= \left(-\int_\Omega \nabla\cdot(V\rho)\,dx\right)\Delta t \quad [\text{integration by parts}]$$

We expand: $\rho(x,t+\Delta t) - \rho(x,t) = \Delta t\,\rho_t(x,t) + O(\Delta t^2)$

to get $$\int_\Omega \left[\rho_t + \nabla\cdot(V\rho)\right] dx = O(\Delta t).$$

Taking the limit $\Delta t \to 0$ and $\Omega \to \{x\}$, we then get the <u>continuous model</u>:

$$(4) \qquad \begin{cases} \rho_t + \nabla \cdot (\rho \vec{v}) = 0 \\ v = K * \rho \end{cases}$$

where $K(z) = F(|z|) \frac{z}{|z|}$ and

$K * \rho = \int_{\mathbb{R}^n} K(x-y) \rho(y)\, dy$ is the convoluti

<u>Stability of constant states</u>:

Since $K * 1 = 0$, $\rho \equiv$ const is a steady state of (4). The distinction between a spreading and a catastrophic regime can be explained in terms of linear stability of a constant density state. In particular, the necessary condition for spreading density is that $\rho = 1$ is <u>stable</u> w.r.t. linear perturbations; the necessary condition for catastrophic regime is that $\rho = 1$ is <u>unstable</u> w.r.t. linear perturbatio

So we let $\rho(x,t) = 1 + \varphi(x,t)$; $\varphi \ll 1$;

then $\rho V = (1+\varphi)\, K*(1+\varphi) = K*\varphi + O(\varphi^2)$

Setting $\varphi(x,t) = e^{\lambda t}\varphi(x)$ we get the <u>linearized equations</u>:

(5) $$\lambda\varphi + K*\varphi = 0 .$$

Next, we make an <u>anzatz</u>: $\varphi(x) = e^{i\vec{m}\cdot\vec{x}}$, $i \equiv \sqrt{-1}$

and compute: $$K*\varphi = e^{i\vec{m}\cdot\vec{x}} \int e^{-m\cdot y i} F(|y|)\frac{y}{|y|}\,dy$$

so that $$\boxed{\lambda = +i\vec{m}\cdot \int e^{-m\cdot y i} F(|y|)\frac{y}{|y|}\,dy}$$

Next, consider the case of <u>small frequency</u>, $\vec{m} \ll 1$.

Then $e^{-m y i} = 1 - m\cdot y i$ and using $K*1 = 0$

we get: $$\int e^{-m y i} F(|y|)\frac{y}{|y|}\,dy = -i\int m\cdot y \frac{F(|y|)}{|y|}\, y\,dy .$$

In the case of <u>1-D</u>, this becomes:

$$\int_{-\infty}^{\infty} m y\, F(|y|)\frac{y}{|y|}\,dy = 2m\int_0^{\infty} y F(y)\,dy .$$

For the <u>morse force</u>, $F(|y|) = e^{-|y|} - G e^{-|y|/L}$

we then obtain:

$$\int_0^\infty y F(y) = 1 - GL^2$$

so that $\boxed{\lambda = 2m^2(1 - GL^2)}$

thus, in 1-D, the spreading will happen only if $GL^2 < 1$

otherwise, if $GL^2 > 1 \Rightarrow$ <u>catastrophic regime</u>.

In 2-D, let $y = r(\cos\theta, \sin\theta)$ so that

$$\int_{\mathbb{R}^2} m \cdot y \frac{F(|y|)}{|y|} y\, dy = \int_0^{2\pi}\int_{r=0}^{\infty} r(m_1 \cos\theta + m_2 \sin\theta) \frac{F(r)}{r} r(\cos\theta, \sin\theta)\, r\, dr\, d\theta$$

$$= \Big(m_1 \underbrace{\int_0^{2\pi} \cos^2\theta\, d\theta}_{\pi},\ m_2 \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}_{\pi}\Big) \underbrace{\int_0^\infty F(r) r^2\, dr}_{2 - 2GL^3}$$

$$\Rightarrow \boxed{\lambda = 2\pi(m_1^2 + m_2^2)(1 - GL^3)}$$

<u>Conclusion</u>: The catastrophic swarm for the Morse force forms if $0 < G < 1$, $L > 1$ and $\boxed{GL^2 > 1 \text{ [in 1-D] or } GL^3 > 1 \text{ [in 2D]}}$

Explicit solutions to aggregation model:

Next, consider the case

$$(6) \qquad F(r) = \begin{cases} 1 - r & \text{in } 1\text{-D} \\ \frac{1}{r} - r & \text{in } 2\text{-D.} \end{cases}$$

We claim: the steady state to (4) with $F(r)$ given by (6) is given by:

$$(7) \qquad \rho(x) = \begin{cases} 1, & x \in B_1(0) \\ 0, & x \notin B_1(0) \end{cases}$$

where $B_1(0) = \{x : |x| < 1\}$.

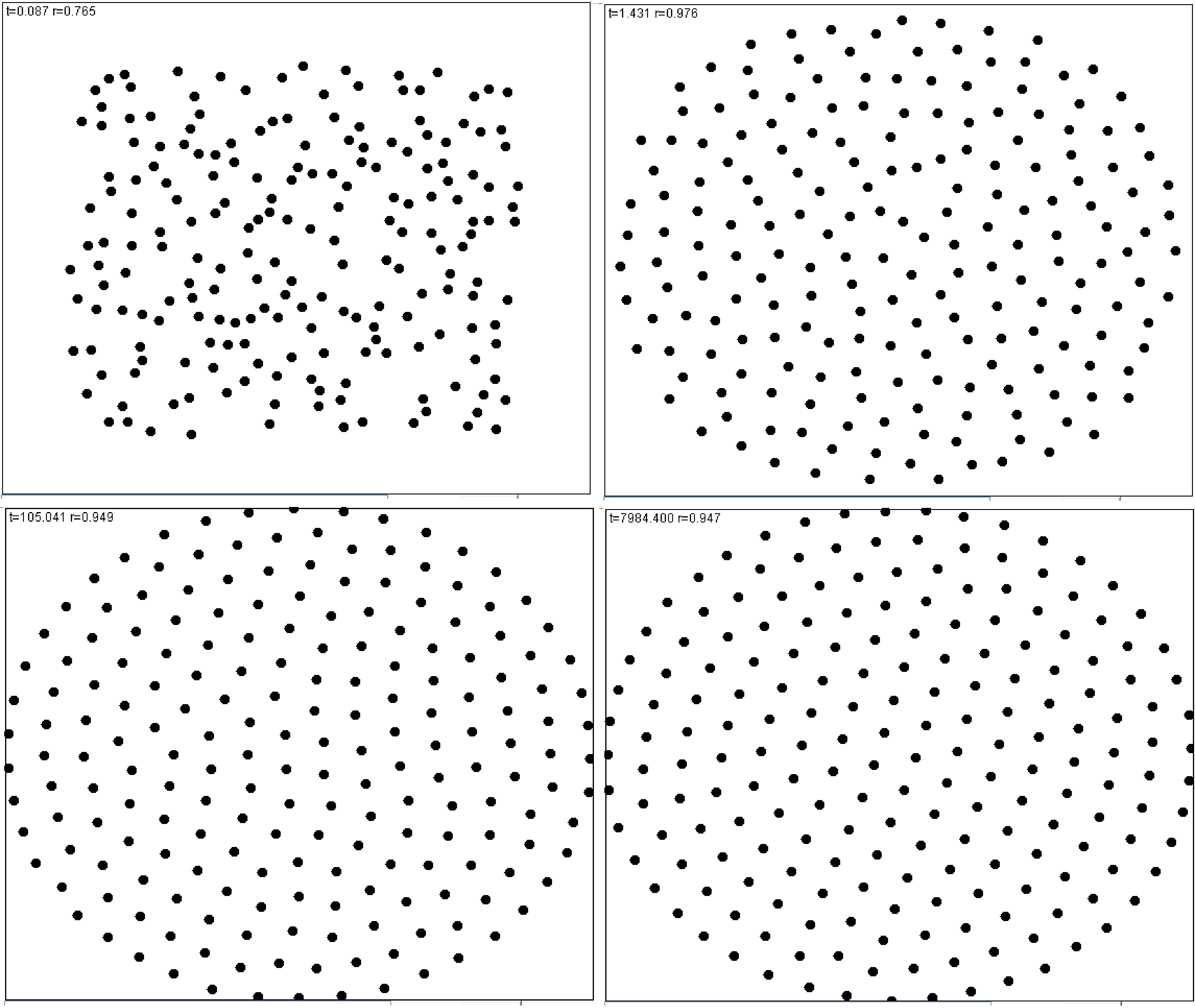
Proof: Let $G(r) = \int F = \begin{cases} r - \frac{r^2}{2}, & \text{in } 1\text{-D} \\ \ln r - \frac{r^2}{2}, & \text{in } 2\text{-D.} \end{cases}$

We have: $v = \int \nabla_x G(|x-y|) \rho(y)\, dy$.

Now define $\omega(x) = \int G(|x-y|)\rho(y)\, dy$.

Consider the 2-D case first. Then

$$\Delta_x G = 2\pi \delta(x-y) - 2$$



Constant density-state in 2D, $F(r)=1/r-r$; N=200 particles.

So that
$$\Delta w = 2\pi \rho(x) - 2M .$$
where
$$M = \int_{\mathbb{R}^2} \rho(x)\, dx .$$

We seek a steady state with the property that
$$\begin{cases} \rho = 0; \ \vec{v} \neq 0 & \text{outside } B_L(0) \\ \rho \neq 0; \ \vec{v} = 0 & \text{inside } B_L(0) \end{cases}$$
where $L > 0$ is some constant to be found.

Then $\nabla \cdot V = 0 = \Delta w$ inside $B_L(0)$
$$\Rightarrow \rho(x) = \begin{cases} \frac{M}{\pi}, & x \in B_L(0) \\ 0, & x \notin B_L(0) \end{cases}$$

~~So~~ Moreover, $M = \int \rho = \pi L^2 \left(\frac{M}{\pi}\right) \Rightarrow \boxed{L = 1}$

This proves (7).

In 1-D, we have: $\Delta G = 2\,\delta(x-y) - 1$

$$\Rightarrow 2\rho(x) - M = 0 \Rightarrow \rho(x) = \frac{M}{2}, \quad x \in [-L, L]$$

$$\text{and } M = \int_{-L}^{L} \rho(x) = \frac{M}{2}(2L) \Rightarrow L = 1$$

$\blacksquare$

Dynamics for $F(r) = 1 - r$ in 1-D:

We compute: $$v(x) = \int_{-\infty}^{\infty} (1 - |x-y|)\,\mathrm{sign}(x-y)\,\rho(y)\,dy$$

$$= 2\int_{-\infty}^{x} \rho(y)\,dy - M(x+1)$$

[where we assume, WLOG, $\int_{-\infty}^{\infty} y\rho(y)\,dy = 0$

and $M = \int_{-\infty}^{\infty} \rho(y)\,dy$]

So <u>define</u> $\quad \omega(x) = \int_{-\infty}^{x} \rho(y)\,dy \; ;$ $\qquad$ (8)

then $\quad V = 2\omega - M(x+1)$

and $\quad V_x = 2\rho - M$

Now $\quad \rho_t + (V\rho)_x = 0$

$$\Rightarrow \rho_t + V\rho_x = -V_x \rho \qquad (9)$$

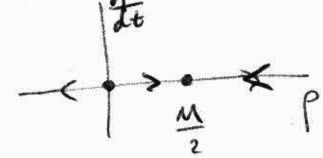
<u>Characteristic curves:</u> We ~~Ro~~ define the characteristics $X(t; x_0)$ to be sol'n to:

$$\begin{cases} \frac{d}{dt} X(t; x_0) = V \\ X(0, x_0) = x_0 \end{cases}$$

Then along X, we have: $\begin{cases} \frac{d}{dt}\rho = -V_x \rho \\ \rho(0; x_0) = \rho_0(x_0) \end{cases}$

where $\rho_0(x_0)$ is the initial density distribution

<u>Thus</u>: $\quad \boxed{\frac{d}{dt}\rho = \rho(M - 2\rho)}$



so that $\quad \boxed{\rho \to \frac{M}{2} \quad \text{as } t \to \infty}$

$\left[\text{explicitly,} \quad \rho = \dfrac{M}{2 + e^{-t}\left(-2 + \frac{M}{\rho_0}\right)} \right]$

From (9) we have:

$$w_t + V\rho = 0 \Rightarrow w_t + V w_x = 0.$$

Thus, w is <u>constant</u> along the characteristics

$$\Rightarrow \quad w = w_0(x_0) = \int_{-\infty}^{x_0} \rho_0(\tau)\, d\tau$$

we get: $\begin{cases} \frac{d}{dt} X = 2w_0 - M(X+1) \\ X(0; x_0) = x_0 \end{cases}$

$$\Rightarrow \boxed{\begin{cases} X = e^{-Mt} x_0 + \frac{1}{M}\left(1 - e^{-Mt}\right)\left(2W_0(x_0) - M\right) \\ \rho = \frac{M}{2 + e^{-t}\left(-2 + \frac{M}{\rho_0(x_0)}\right)} \end{cases}}$$

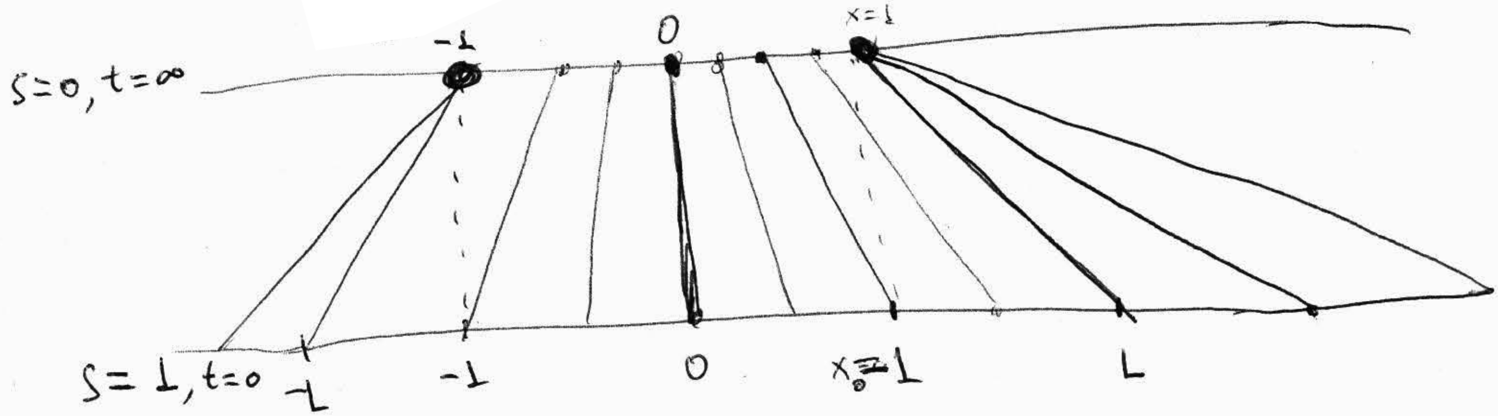
<u>Example:</u> Suppose $\rho_0(x) = \begin{cases} 1, & x \in [-L, L] \\ 0, & x \notin [-L, L]. \end{cases}$

Then $\int_{-\infty}^{\infty} x\rho(x) = 0$ and $M = 2L$ and

$$X = \begin{cases} -1 + e^{-2Lt}(x_0 + 1), & x_0 < -L \\ \frac{x_0}{L} + e^{-2Lt}\left(x_0\left(1 - \frac{1}{L}\right)\right), & x_0 \in [-L, L] \\ 1 + e^{-2Lt}(x_0 - 1), & x_0 > L \end{cases}$$

$$\rho = \begin{cases} 0, & |x_0| > L \\ \frac{L}{1 + e^{-t}(L-1)}, & |x_0| < L \end{cases}$$

Letting $s \equiv e^{-2Lt}$, the graph of characteristics looks like:



- Characteristics intersect at $t=\infty$, $x=\pm 1$

- Note also that if $L=1$ then we recover (7)!
- We get:

$$\rho(x) = \begin{cases} 0, & |x| > R(t) \\ \dfrac{L}{1+e^{-t}(L-1)}, & |x| < R(t) \end{cases}$$

where $R(t) = 1 + e^{-t}(L-1)$