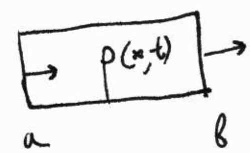
Traffic Model

Continuous model: Let ρ denote the density of cars and let $q(x,t)$ denote the <u>flux</u> i.e. rate of change of cars:

Then



$$\frac{d}{dt}\int_a^b \rho\, dx = q(a,t) - q(b,t)$$

Let $u(x,t)$ denote the velocity of cars; then

$$q = \rho u$$

Now taking $b = a + \Delta x$; $\Delta x \ll 1$

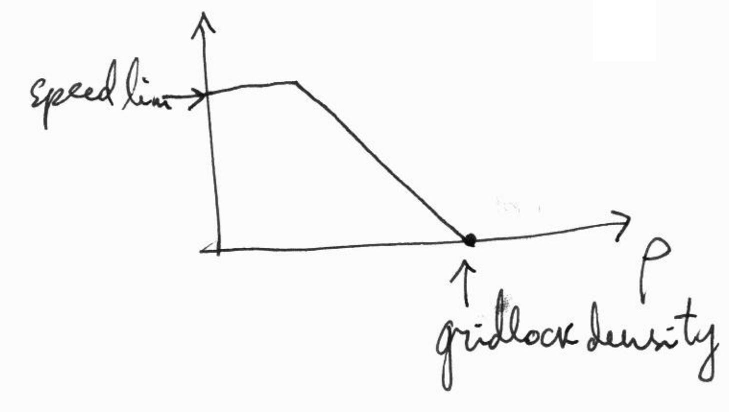
we obtain : $\frac{d}{dt}\left(\Delta x\, \rho(a)\right) \sim q(a,t) - q(a+\Delta x, t)$

Taking the limit, we get; after setting $a = x$:

$$\boxed{\frac{d}{dt}\rho(x,t) + \frac{d}{dx}\Big(\rho(x,t)\, u(x,t)\Big) = 0}$$

<u>Assume</u> $u = u(\rho)$ [cars respond locally]

The curve $u = u(\rho)$ looks like this (maybe):



$$\rho_t + \frac{\partial}{\partial x}(\rho u) = 0 \; ; \quad u = u(\rho)$$

$\uparrow$ mass conservation $\qquad$ $\uparrow$ Eq'n of state

<u>Simple example:</u> $u(\rho) = c$ [velocity indep. of car density]

$\Rightarrow \quad \rho_t + c\,\rho_x = 0 \; ; \quad \rho(x,0) = \rho_0(x)$ [initial density]

<u>Method of Characteristics:</u>

<u>Def:</u> Characteristics are curves in the x-t plane along which $\rho \equiv$ const.

Label such curves as $x = x(t; x_0)$

with $\rho = \rho_0(s)$ along $x(t, x_0)$

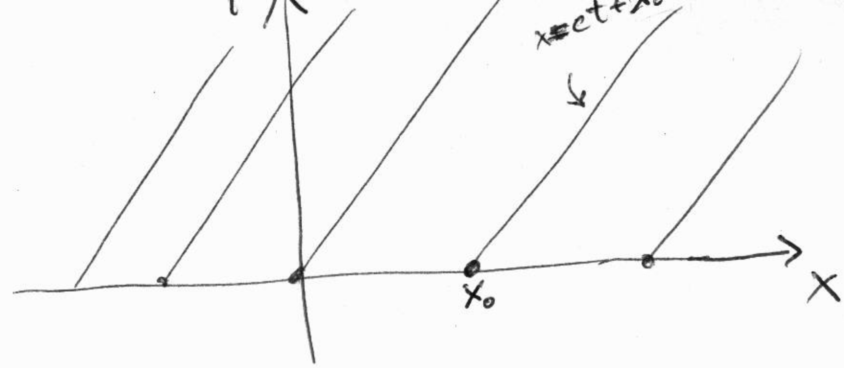
<u>Then</u> $\quad \frac{d}{dt}\rho(t,x) = \rho_t + \rho_x\, x_t = 0$

$\Rightarrow$ Choose $\boxed{x_t = c} \Rightarrow \begin{cases} x_t(t; x_0) = c \\ x(0, x_0) = x_0 \end{cases}$

where s is a parametrization of i.c.

$\Rightarrow \begin{cases} x = ct + x_0 \\ \rho = \rho_0(x_0) \end{cases} \Rightarrow x_0 = x - ct$

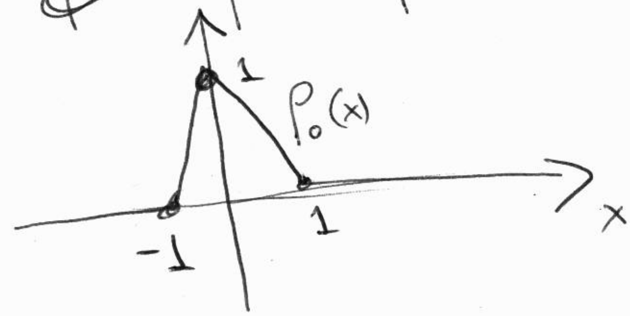
$\Rightarrow \boxed{\rho(t,x) = \rho_0(x - ct)}$



<u>Check</u>: $\cdot\ \rho(x,0) = \rho_0(x - c\cdot 0) = \rho_0(x)$ ✓

$\cdot\ \rho_t + c\rho_x = -c\rho_0' + c\rho_0' = 0$ ✓

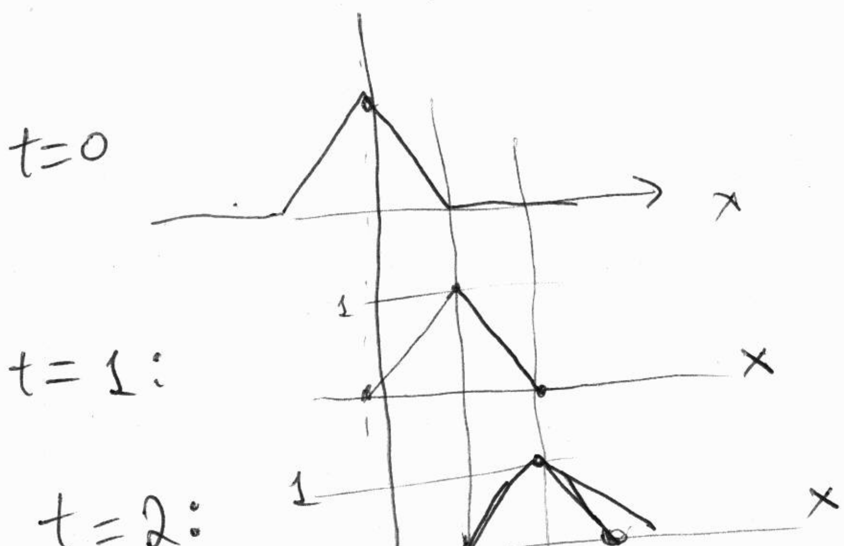
<u>Ex</u>: Take $\rho_0 \equiv$



Also take $c = 1$.

Then $\rho(x,1) = \rho_0(x-1)$

$\rho(x,2) = \rho_0(x-2)$



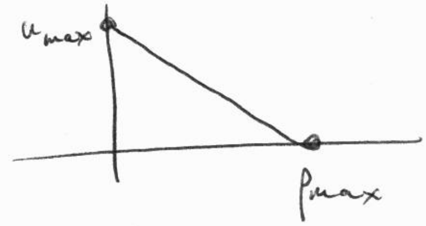
$\Rightarrow$ hump is moving to the right wit velocity $c = 1$

Density-dependent velocity:

We consider a more realistic eq'n of stat of the form,

$$u = u(\rho) = \frac{u_{max}}{\rho_{max}}(\rho_{max} - \rho)$$

where $u_{max} \equiv$ max speed
achieved at low density
$\rho_{max} \equiv$ "gridlock" density



Eg. take $\boxed{u = 1 - \rho}$; we get

$$\rho_t + (\rho(1-\rho))_x = 0$$

$$\boxed{\rho_t + (1-2\rho)\rho_x = 0}$$

As before, we seek characteristic curves $x = x(t; x_0)$ along which $\rho = \rho_0(x_0)$ is const. Then we get:

$$\frac{d}{dt}(\rho) = \rho_t + x_t \rho_x = 0 \Rightarrow \boxed{x_t = 1 - 2\rho}$$

Thus we get: $\begin{cases} \frac{d}{dt} x = 1 - 2\rho \\ \frac{d}{dt}\rho = 0 \end{cases}$ and $\begin{cases} x = x_0 & \text{at } t = 0 \\ \rho = \rho_0 & \text{at } t = 0 \end{cases}$

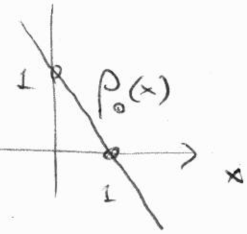
Thus:

$$\boxed{\begin{cases} x = (1-2\rho_0)t + x_0 \\ \rho = \rho_0(x_0) \end{cases}}$$

Example: Suppose that $\rho_0(x) = 1 - x$

Then

$$\begin{cases} x = (1-2(1-x_0))t + x_0 = 2x_0 t - t + x_0 \\ \rho = 1 - x_0 \end{cases}$$



$$\Rightarrow \quad x_0 = \frac{x + t}{1+2t} \ ; \quad \rho = 1 - x_0$$

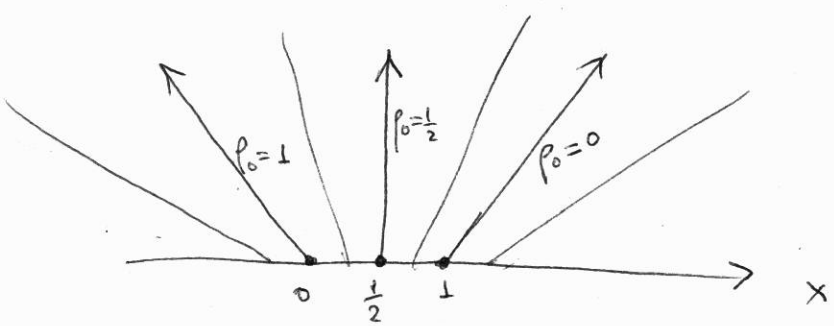
$$\Rightarrow \quad \boxed{\rho(x,t) = 1 - \frac{x + t}{1+2t}}$$

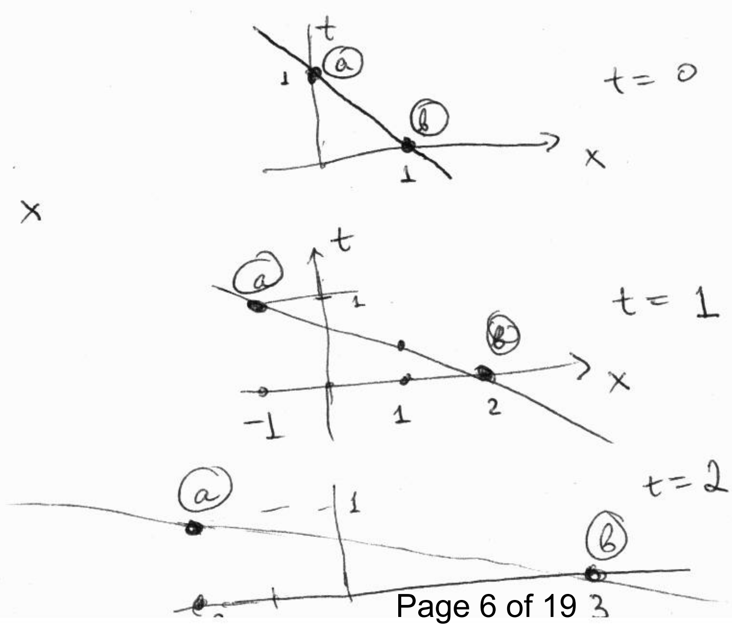
Sketch of characteristics:

$x_0 = 0 \Rightarrow \rho_0 = 1 \Rightarrow x = -t$

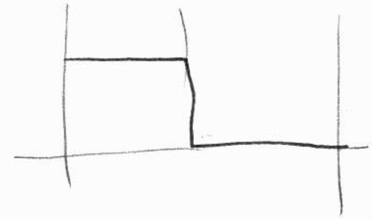
$x_0 = 1 \Rightarrow \rho_0 = 0 \Rightarrow x = 1 + t$

$x_0 = \frac{1}{2} \Rightarrow \rho_0 = \frac{1}{2} \Rightarrow x = \frac{1}{2}$





<u>Ex 2</u>: Suppose $\rho_0 = \begin{cases} 1 & , \quad x < 0 \\ 0 & , \quad x > 0 \end{cases}$



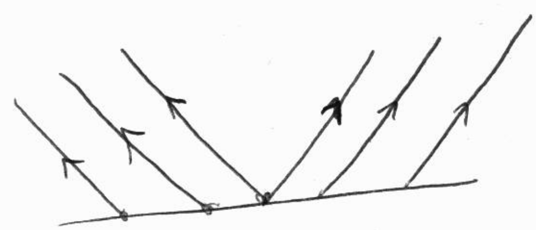
This corresponds to the "green light" problem: the light just turned green at $t=0$, $x=0$.

As before: $$x = (1 - 2\rho_0(x_0))\, t + x_0$$
$$\rho = \rho_0(x_0)$$

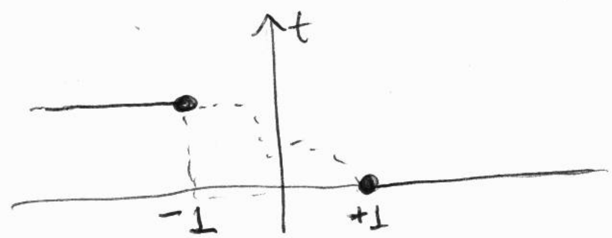
If $x_0 < 0$ then $\begin{cases} x = -t + x_0 \\ \rho = 1 \end{cases}$

If $x_0 > 0$ then $\begin{cases} x = t + x_0 \\ \rho = 0 \end{cases}$

Characteristic curves are:



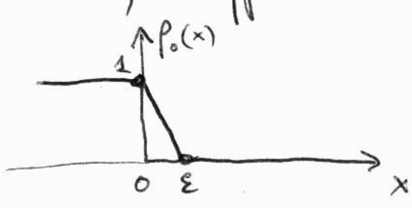
At $t=1$, sol'n is:



$$\rho = \begin{cases} 1, & x < -1 \\ 0, & x > 1 \end{cases}$$

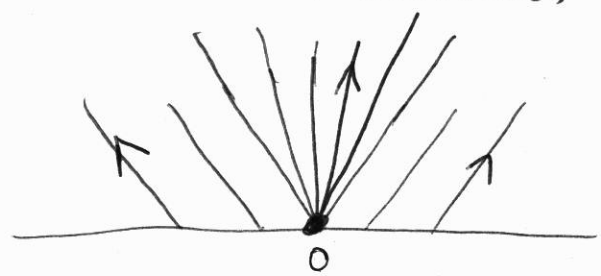
Note that sol'n is undefined ~~at~~ for $x \in (-t, t)$!

- This is due to discontinuity of initial cond. at $x_0 = 0$.
- To overcome this problem, approximate i.c. by

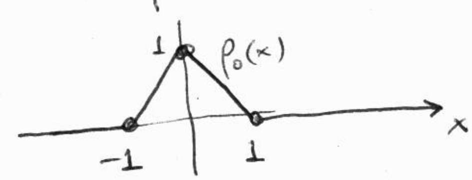
$\rho_0 =$  ; let $\varepsilon \to 0$.

Then in limit $\varepsilon \to 0$, the characteristics look like:

Characteristics coming from the origin are called <u>expansion fan</u>.



<u>Shock formation</u>: Consider $\begin{cases} \rho_t + (1-2\rho)\rho_x = 0 \\ \rho_0 = \end{cases}$



<u>Characteristics</u>:

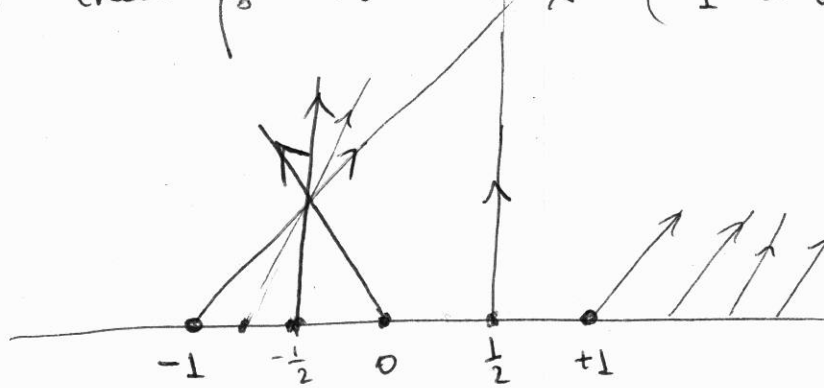
If $x_0 > 1 \Rightarrow \quad x = t + x_0$

If $x_0 < -1 \Rightarrow \quad x = t + x_0$

If $x_0 \in [0,1]$ then $\rho_0 = 1 - x_0 \Rightarrow x = (-1 + 2x_0)t + x_0$

If $x_0 \in [-1,0]$ then $\rho_0 = x_0 + 1 \Rightarrow x = (-1 - 2x_0)t + x_0$

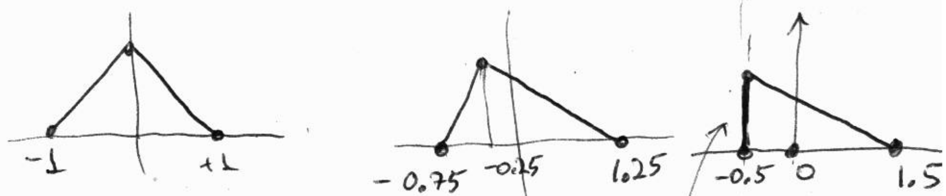
<u>Sketch</u>:



<u>Note</u>: Characteristics intersect at $t = \frac{1}{2}$!!

- This indicates the presence of a shock formation at time $t = \frac{1}{2}$!

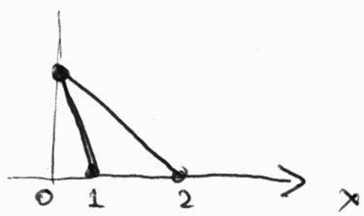
Sketch:



$t = 0$ $\qquad$ $t = 0.25$ $\qquad$ $t = \frac{1}{2}$

Shock forms!

- After $t = 0.5$, the solution becomes mult-valued!!



$t = 1$

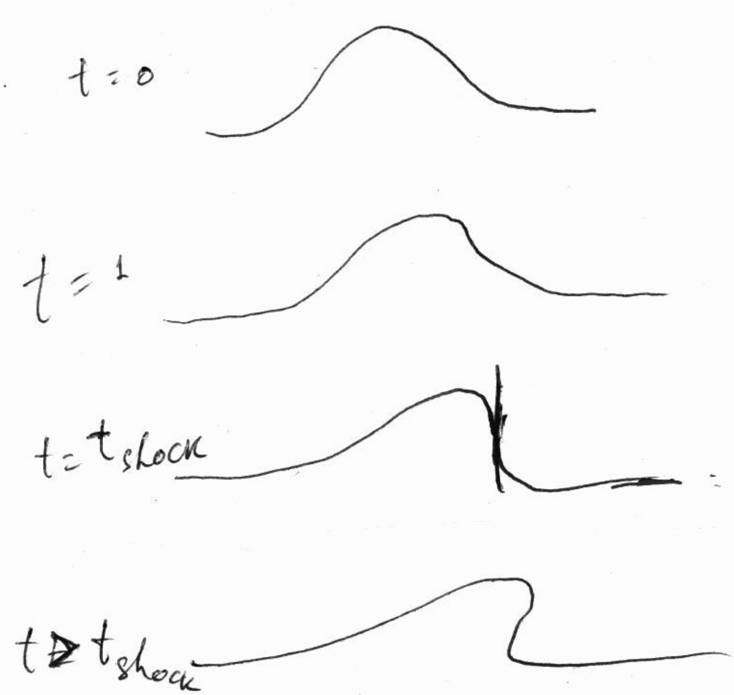
- Models needs to be modified for $t > \frac{1}{2}$.

Ex Consider $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \varphi(x) := e^{-x^2}. \end{cases}$

Sol'n along characteristics:

$$\begin{cases} x = \varphi(x_0)\, t + x_0 \\ \varphi = \varphi(x_0) \end{cases}$$

- The speed is $c = u$ and is increasing with u.
- Schematically, sol'n looks like:



[the top of bump moves faster than its bottom]

To compute t_{shock}: At the onset of the shock formation, we have $\frac{dx}{d\varphi} = 0$, $\frac{d^2x}{d\varphi^2} = 0$

for some $x = x_{shock}$, $t = t_{shock}$.

We compute:
$$\frac{dx}{d\varphi} = \frac{\left(\frac{dx}{dx_0}\right)}{\left(\frac{d\varphi}{dx_0}\right)} = \frac{\varphi'(x_0)t+1}{\varphi'(x_0)} = 0$$

and
$$\frac{d^2x}{d\varphi^2} = \frac{\frac{d}{dx_0}\left(\frac{\varphi'(x_0)t+1}{\varphi'(x_0)}\right)}{\varphi'(x_0)} = -\frac{\varphi''(x_0)}{(\varphi'(x_0))^3} = 0$$

$$\Rightarrow \begin{cases} \varphi'(x_0)t+1 = 0 \\ \varphi''(x_0) = 0 \end{cases}$$

Ex If $\varphi(x_0) = e^{-x^2}$ then $\varphi' = -2x_0 e^{-x_0^2}$

$$\varphi'' = (+4x_0^2 - 2)e^{-x_0^2}$$

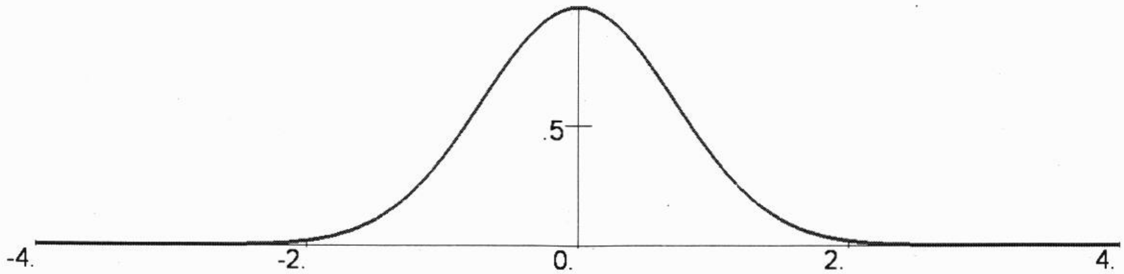
so $\varphi'' = 0 \iff \boxed{x_0 = \frac{1}{\sqrt{2}}}$

and $t = -\dfrac{1}{\varphi'(x_0)} = \dfrac{1}{\sqrt{2}} e^{\frac{1}{2}} = 1.1658$

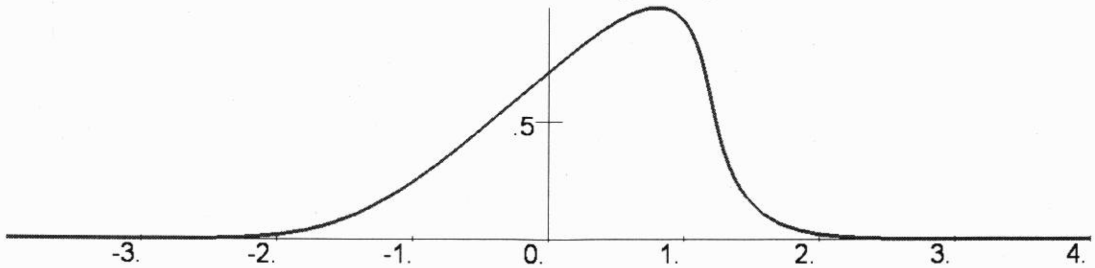
and $x = e^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}} e^{\frac{1}{2}}\right) + \frac{1}{\sqrt{2}} = \sqrt{2}$

So shock (first) occurs at
$$\boxed{\begin{cases} t_{shock} = 1.1658 \\ x_{shock} = \sqrt{2} \end{cases}}$$

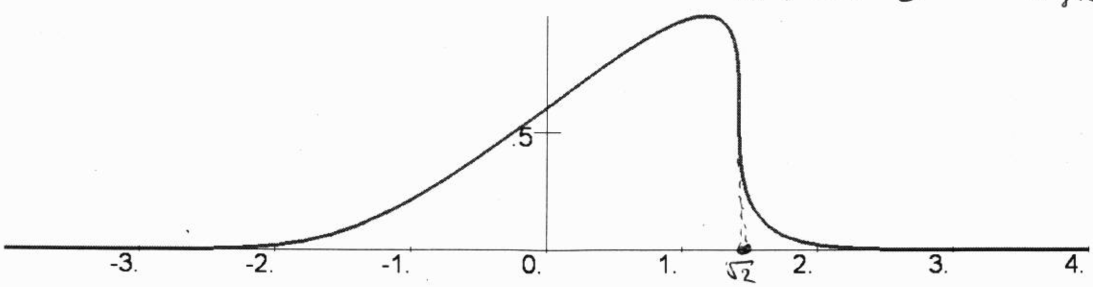
$t = 0$



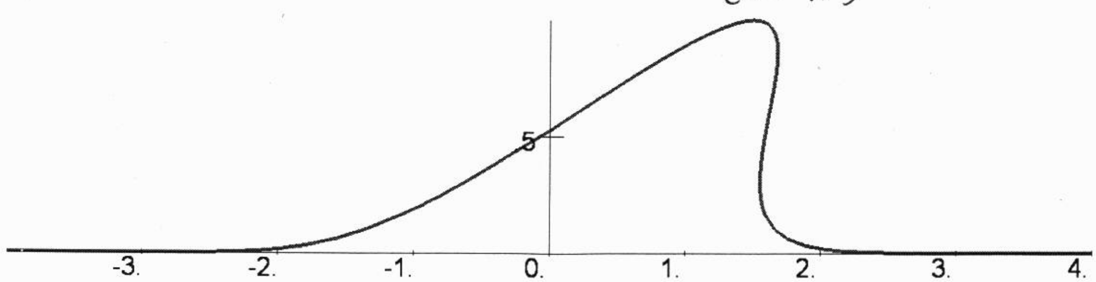
$t = 0.8$



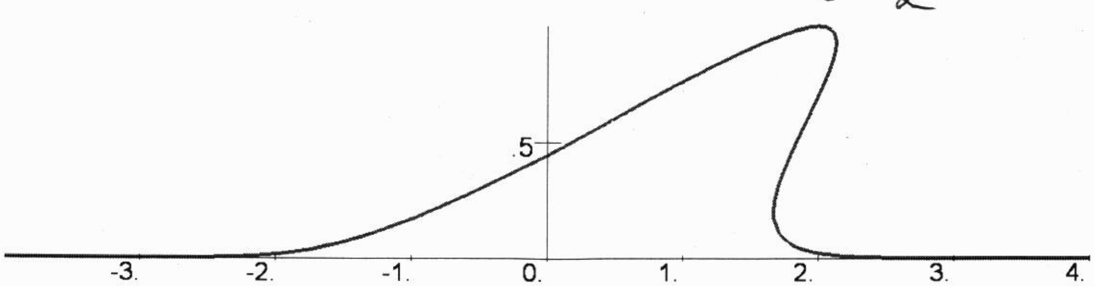
$t = 1.1658 = t_{shock}$



$t = 1.5$



$t = 2$



Q: What happens for $t > t_{shock}$?

- The basic model needs to be modified
- Incorporate the effect of "looking ahead" into the equation of state:

Take $q = u\rho - a\rho(x+h)$, $h \ll 1$; $u = 1-\rho$ as before.

Then $q = (1-\rho)\rho - a\rho - ah\rho_x$

$q_x = \left((1-a) - 2\rho\right)\rho_x - ah\rho_{xx}$

and $\rho_t + q_x = 0$

$\Rightarrow$ Let $u = (1-a) - 2\rho$; $\varepsilon = ha$

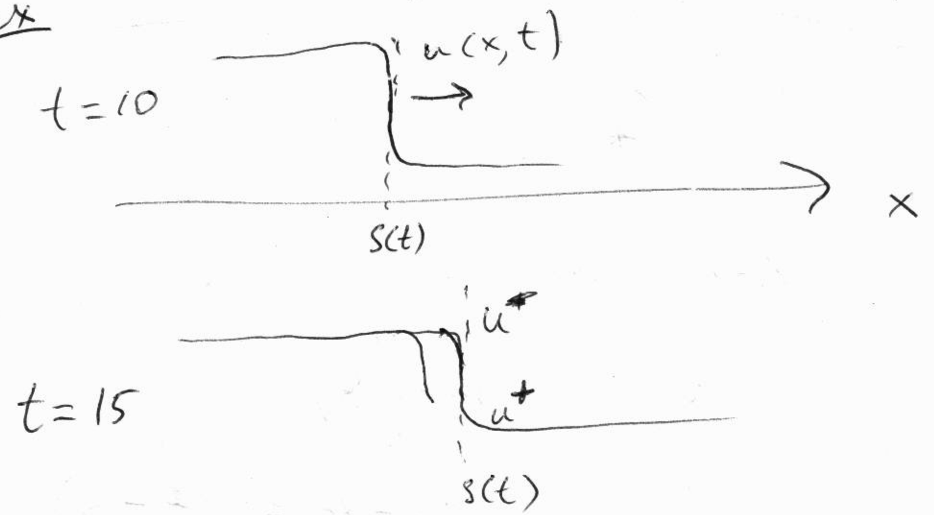
then
$$\boxed{\begin{cases} u_t + u u_x = \varepsilon u_{xx} \\ u(x,0) = \varphi(x) \end{cases}} \qquad (B)$$

This is Burger's equation.

- The small diffusion has the effect of "smoothing the shocks" so that the solution is well-defined for all time.

Motion of the shock: Let $x = s(t)$ be the position of the shock at time t; due to the εu_{xx} term in (B), an interface layer forms at the shock:

Ex



In the inner layer change variables:

Let $x = s(t) + \varepsilon y$, $\quad y = \dfrac{x - s(t)}{\varepsilon}$ and

$$u(x,t) = \mathcal{U}(y) = \mathcal{U}\left(\frac{x - s(t)}{\varepsilon}\right)$$

Then $\quad -s'(t)\,\mathcal{U}_y + \mathcal{U}_y\,\mathcal{U} = \mathcal{U}_{yy}$

$$\Rightarrow \quad \mathcal{U}_y = \frac{\mathcal{U}^2}{2} - s'(t)\,\mathcal{U} + A(t)$$

To determine $A(t)$, let

$$u^{\pm}(t) = \lim_{x \to s(t)^{\pm}} u(x,t) \quad ; \qquad \text{then}$$

We assume $\mathcal{U}_y \to 0$ as $y \to \pm\infty$

and $\quad \mathcal{U} \to u^{\pm}$ as $y \to \pm\infty$.

Thus we get: $\begin{cases} \dfrac{u_+^2}{2} - s' u_+ = A \\ \dfrac{u_-^2}{2} - s' u_- = A \end{cases}$

$$\Rightarrow \quad A = \frac{u_- u_+}{2} \; ; \qquad \boxed{s'(t) = \frac{u_+ + u_-}{2}}$$

This eq'n determines the motion of the front.

<u>Interface profile</u>: $2U_y = (U - u_-)(U - u_+)$

$$\Rightarrow \quad U(y) = \frac{u_-}{2} + \left(u_+ - \frac{u_-}{2}\right) \tanh\left(y\left(\frac{u_- - u_+}{4}\right)\right)$$

<u>Ex</u>: Suppose the initial condition in (B)

is $\varphi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$.

Then the shock forms immediately (at $t = 0$); with $u_+ = 1$, $u_- = 0$ $\forall t$;

$$\Rightarrow \quad \boxed{s(t) = \tfrac{1}{2} t}$$

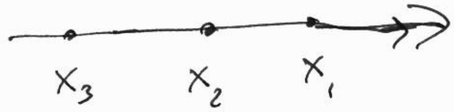
and $U(y) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{y}{4}\right)$

$$\Rightarrow \quad \boxed{u(x,t) \approx \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)}$$

Car-following model:

$$\ddot{x}_n = -\lambda(\dot{x}_n - \dot{x}_{n-1}), \quad \lambda > 0$$

- If $\dot{x}_n < \dot{x}_{n-1} \Rightarrow$ speed up
  $\dot{x}_n > x_{n-1} \Rightarrow$ slow down.



Integrating we get:

$$\dot{x}_n = -\lambda(x_n - x_{n-1}) + d_n \quad \text{for some const. } d_n.$$

<u>Simplest situation</u>: suppose all cars are moving with the same speed and are <u>equidistantly spaced</u>:

$$\dot{x}_n = \dot{x}, \quad d_n = d; \quad x_{n-1} - x_n = \text{constant}$$

Now $\rho \sim \dfrac{\#\text{cars}}{\text{unit length}} = \dfrac{1}{x_{n-1} - x_n}$

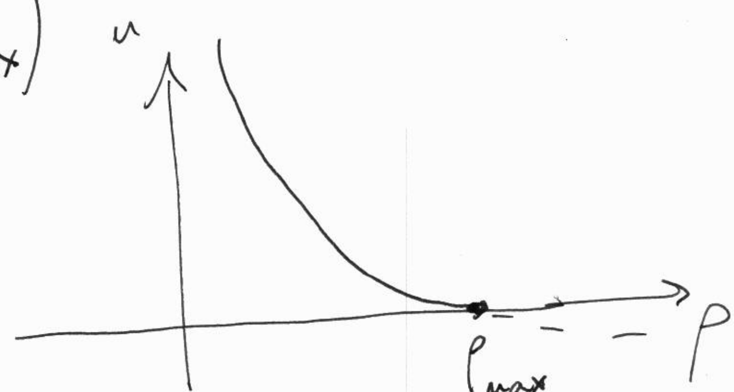
$\Rightarrow x_n - x_{n-1} \sim -\dfrac{1}{\rho}$

So if $u \equiv \dot{x}_n$ is the velocity field then

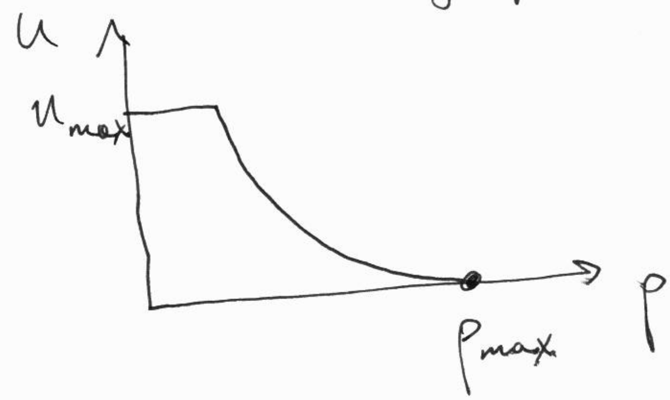
$$\boxed{u \sim \frac{\lambda}{\rho} + d}$$

Now ~~also~~ assume that $u = 0$ at some $\rho = \rho_m$ where ρ_m is the "bumper-to-bumper" density ; then $d = -\frac{\lambda}{\rho_{max}}$;

$$u = \lambda \left(\frac{1}{\rho} - \frac{1}{\rho_{max}} \right)$$



Remark: more realistic graph would be



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Car-following with lag:

$$\ddot{x}_n(t+\tau) = -\lambda\left(\dot{x}_n - \dot{x}_{n-1}\right)$$

Stability

- Local stability: Suppose all cars travel with speed $\dot{x}_n = u$

Now suppose the lead car keeps its speed and the following car speed is perturbed: $V_n = u + y$; then

$$\dot{y}(t+\tau) = -\lambda y$$

$\Rightarrow$ • Hopf when $\tau\lambda = \frac{\pi}{2}$

at $\tau = \tau_h = \frac{\pi}{2\lambda}$;

• stable if $\tau < \tau_h$.

Asymptotic stability:

Scale, $u = 1$; suppose that

$$v_n = u + e^{\cancel{\sigma} t} f_n$$

$$\sigma \cancel{\sigma} e^{\cancel{\sigma\tau}} f_n = -\lambda\left(f_n - f_{n-1}\right)$$

$$f_n = f_{n-1} \frac{\lambda}{\lambda + \sigma e^{\cancel{\sigma}\tau}}$$

$$= \underbrace{\left(\frac{1}{1 + \frac{\sigma}{\lambda} e^{\cancel{\sigma}\tau}}\right)}_{\text{amplification factor}} f_{n-1}$$

$$\left|1 + \frac{\sigma}{\lambda} e^{\sigma\tau}\right| > 1$$

stable

Oscillations: if $\sigma = i\omega$ then

$$\left|1 + \frac{\sigma}{\lambda} e^{\sigma\tau}\right|^2 = \left(1 + \frac{i\omega}{\lambda} e^{+i\omega\tau}\right)\left(1 - \frac{i\omega}{\lambda} e^{-i\omega\tau}\right)$$

$$= 1 + \frac{\omega^2}{\lambda^2} - \frac{2\omega}{\lambda} \sin\omega\tau > 1$$

$$\Longleftarrow \boxed{\lambda\tau < \frac{1}{2}} \Longleftarrow \quad 2\frac{\cancel{\omega}}{\lambda} \sin\omega\tau \le \frac{\omega^2}{\lambda^2} \Leftrightarrow \boxed{\frac{\sin\omega\tau}{\omega} \le \frac{1}{2\lambda}}$$