Klausmeier Model of Vegetation

- In the near-desert conditions, certain vegetation grows in "patches"
- Presumabley this allows the plants to conserve water and survive in near-drought state.

Model using <u>reaction-diffusion system</u>:

- $n(x,t)$: plant density in space "x" and time "t"
- $w(x,t)$: water density

<u>Klausmeier model</u>:
[normalized]

$$\underbrace{\frac{\partial n}{\partial t}}_{\text{rate of change of \cancel{water} plants}} = \underbrace{wn^2}_{\text{plant growth}} \underbrace{- \quad n}_{\text{decay}} + \underbrace{\delta \quad \frac{\partial^2 n}{\partial^2 x}}_{\text{plant "diffusion"}}$$

$$b\frac{\partial w}{\partial t} = \underset{\substack{\uparrow \\ \text{water source}}}{a} - \underset{\substack{\uparrow \\ \text{evaporation}}}{w} - \underset{\substack{\uparrow \\ \text{absorbtion by plants}}}{wn^2} + \underset{\substack{\uparrow \\ \text{diffusion}}}{\frac{\partial^2 w}{\partial^2 x}}$$

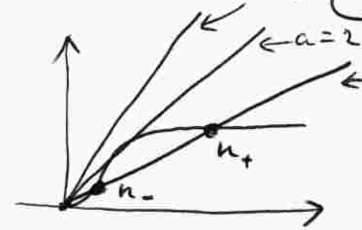
ODE dynamics [Kinetics]

- discard spatial terms:

$$n_t = \omega n^2 - n, \quad b\,\omega_t = a - \omega - \omega n^2$$

- Consider small-b limit [

Then $a - \omega - \omega n^2 \approx 0 \rightarrow \omega = \dfrac{a}{1+n^2} \Rightarrow \boxed{n_t = \dfrac{a n^2}{1+n^2} - n}$



Steady state: $\dfrac{n}{a} = \dfrac{n^2}{1+n^2} = 1 - \dfrac{1}{1+n^2}$

$$n^2 - an + 1 = 0$$

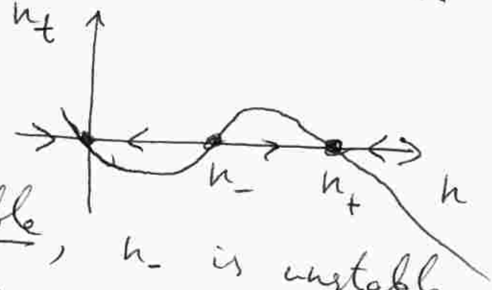
→ Either $n = 0$ or $n = n_\pm = \dfrac{a}{2} \pm \sqrt{\dfrac{a^2}{4} - 1}$

- If $a > 2$, then there are 3 s.s.: $n = n_\pm$ and $n = 0$
- At $a \simeq 2$, the two roots $n_\pm$ merge and dissapear [Fold-point bifurcation]
- Only $n = 0$ ["desert state"] remains if $a < 0$.

Stability [smal-b limit]

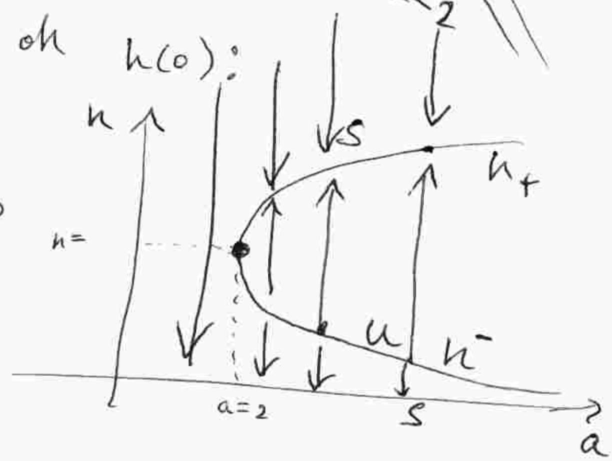
$$n_t = \frac{a n^2}{1+n^2} - n$$

$a > 2$



- If $a > 2$ then n_+, n_0 are stable, n_- is unstable
- If $a \leq 2$ then n_0 is stable and $n_\pm$ do not exist
- Fold-pt bifurcation at $a = 2$

- Eventual steady state depends on $n(0)$:
  - If $a > 2$ and $n(0) > n_-$ then $n(t) \to n_+$ as $t \to \infty$
  - If $a > 2$ and $n(0) < n_-$ then $n(t) \to 0$ as $t \to \infty$ [desert state]
  - If $a < 2$ then $n(t) \to 0$ as $t \to \infty$



Conclusion: In the absence of diffusion [i.e. ODE system] only desert state is possible if $a < 2$

Q: What happens when there is diffusion??

Spatial instability: Linearize around the homogeneous state n_+, ω_+

$$n(x,t) = n_+ + \varphi, \quad \omega = \omega_+ + \psi, \quad \varphi, \psi \ll 1.$$

Write original system in the form $\begin{cases} n_t = \delta n_{xx} + f(n,\omega) \\ 0 = \omega_{xx} + g(n,\omega) \end{cases}$ [where we took $b = 0$]

[with $g(n_+, \omega_+) = 0 = f(n_+, \omega_+)$]

$$\Rightarrow \quad \varphi_t = \delta \varphi_{xx} + f_n(n_+,\omega_+)\varphi + f_\omega(n_+,\omega_+)\psi, \quad \psi_{xx} + g_n\varphi + g_\omega\psi = 0$$

"Separate variables": $\varphi = e^{\lambda t}\hat\varphi$, $\psi = e^{\lambda t}\hat\psi$

$$\Rightarrow \lambda\hat\varphi = \delta\hat\varphi_{xx} + f_n\hat\varphi + f_\omega\hat\psi, \quad \hat\psi_{xx} + g_n\hat\varphi + g_\omega\hat\psi = 0$$

- Plug in $\hat\varphi = \cos(mx)\tilde\varphi$, $\hat\psi = \cos(mx)\tilde\psi \Rightarrow$

$$\Rightarrow \lambda\tilde\varphi = -m^2\delta\tilde\varphi + f_n\tilde\varphi + f_\omega\tilde\psi, \quad -m^2\hat\psi + g_n\tilde\varphi + g_\omega\tilde\psi = 0 \Rightarrow \hat\psi = \frac{g_n\tilde\varphi}{m^2 - g_\omega}$$

$$\Rightarrow \boxed{\lambda = -m^2\delta + f_n + \frac{f_\omega g_n}{m^2 - g_\omega}}$$

Here, we have

$f = \omega n^2 - n$

$g = a - \omega - \omega n^2$

$f_n = 2\omega n - 1 = 1$ $[\omega_+ n_+ = 1]$

$f_\omega = n_+^2$

$\cancel{g_n = -1 - 2\omega n}$

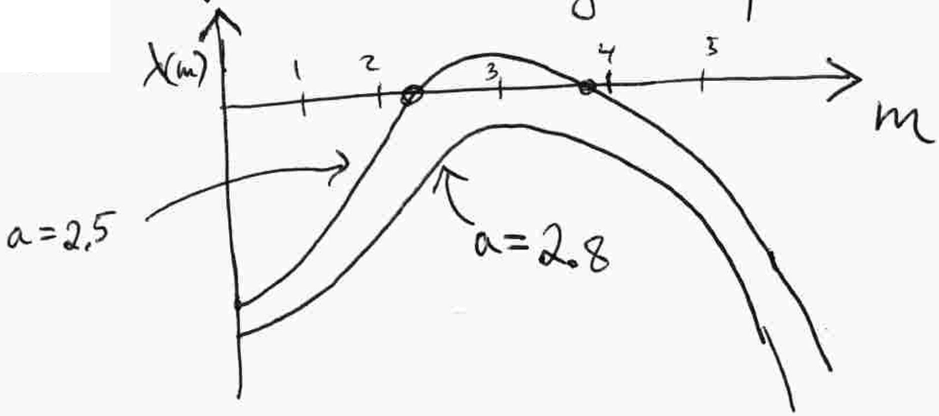
$g_n = -2\omega n = -2$

$g_\omega = -1 - n_+^2 = -a n_+$

$$\Rightarrow \boxed{\lambda(m) = -m^2\delta + 1 + \frac{-2n_+^2}{m^2 + a n_+}}$$

We obtain $\lambda = \lambda(m)$

- Take e.g. $\delta = 0.04$, and plot $\lambda(m)$ for several values of a: Using Maple we get these plots:



- Note that when $a = 2.8$, $\lambda(m) < 0$ for all m. But when $a = 2.5$, there is an unstable band of modes $m \in (2.2, 3.9)$ for which $\lambda(m) > 0$
  - This induces spatial instability!

- Using Maple, we find that the unstable band first appears at $a = a_T \approx 2.632$, $m \approx 3.105$
- This is called "Turing instability".

Numerical experiment:

ex1: - Take $a = 2.7 > a_c$

- Take i.c. $n(x,0) = n_+$ + small random noise
  $w(x,0) = w_+$.
- Take domain $x \in [0, L]$ with $L = 20$; Neuman B.C.
- No inhomogeneities as $t \to \infty$
- See FlexPDE script

ex2: - Take $a = 2.6$, rest as in ex1.

- Inhomogeneities develop, resulting in a pattern with 10 "bumps":

$x=0$ $x=20$

- This corresponds to sol'n of form $\cos\left(10 \cdot \frac{2\pi}{L} x\right)$, $L=20$

$$\approx \cos(3.1\,x)$$

$$\Longleftrightarrow \boxed{m \approx 3.1}$$

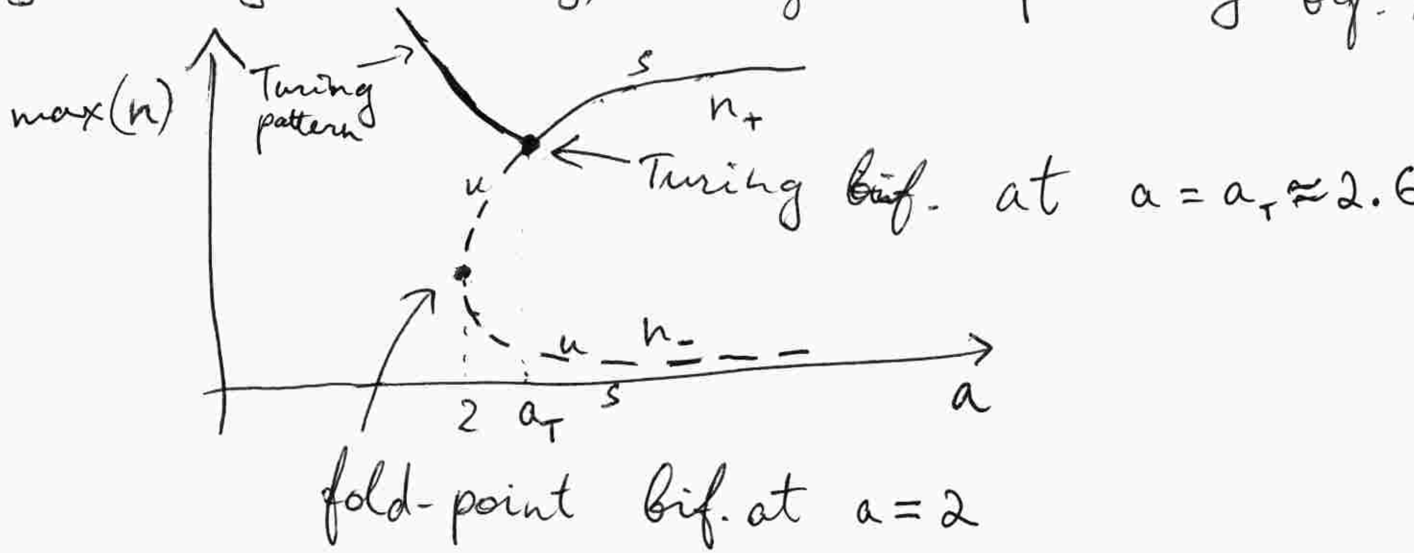
This is the value predicted from the Linear analysis!

Reversability:

- Imagine the parameter "a" [i.e. the precipitation level] is decreased slowly, from say $a=3$ to say $a=1.5$
- ~~If~~ Without the spatial terms, the sol'n hits a fold point at $a=2$, so ~~for $a<2$~~ as a drops below 2, the vegetation collapses to a desert state.
- Since the desert state is stable for all a, if a is increased later on back to 3, the desert state remains: the collapse is irreversible
- However because of Turing instability, if say $\delta=0.0$ the solution becomes inhomogeneous at $a_T \approx 2.6$
  - This turing pattern persists well below $a=$
  - Thus vegetation can survive "in patches" ~~when homo~~ even in the regime where the homogeneous sol'n no longer exists!
  - Moreover, the "patch" pattern is reversable!
  - See FlexPDE script.

- Incorporating Turing instability, we get the following bif. diagram

$\underline{\delta = 0.04}$:



- Turing bif. leads to spatially inhomogeneous patterns
- These patterns extend well beyond the fold-point and allow vegetation to survive in "patches" with limited precipitation
- See FlexPDE script for numerical simulations of this.