A new PDE model of bacterial aggregation

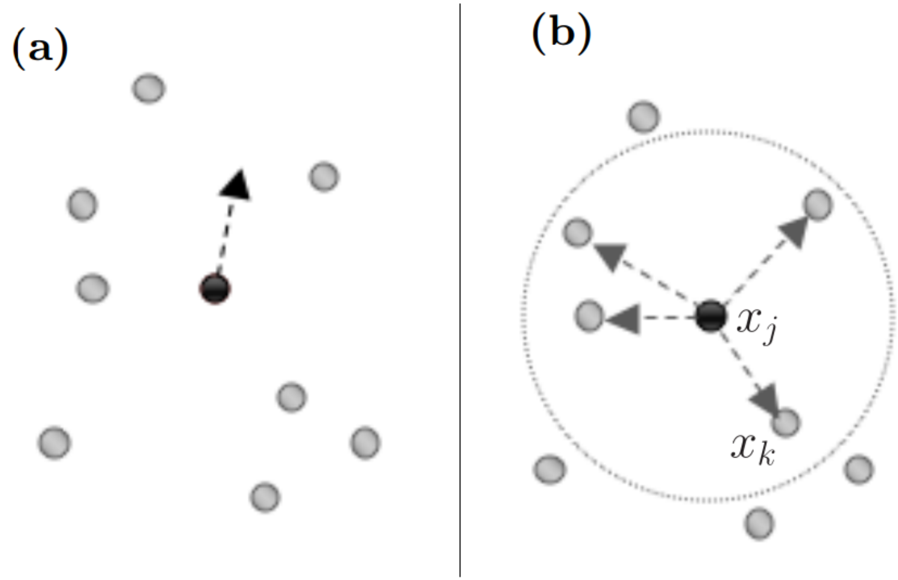
Theodore Kolokolnikov

Joint work with Paul-Christopher Chavy-Waddy

Dalhousie University

Galante-Wisen-Bhaya-Levy model (agent–based):

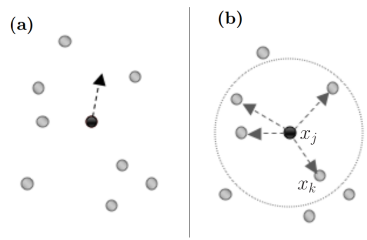
- Each bacteria has position x_j.
- It moves at some rate r. Motion consists of two components: random motion directed motion
  - Random motion: at rate a, choose a direction at random
  - Directed motion: at rate c pick at random another bacteria within its **sensing radius**, and then orients itself to move towards it.



- Aggregations form as switching rate c rate is increased.

- Pseudocode:

```
for t=0:dt:1000
    for j=1:n % cycle through each bacteria
        Initialize $d_j = 0$
        if rand < **a***dt
            Pick a random direction $d_j = \exp(i * \mathrm{rand} * 2pi)$
        end
        if rand < **c***dt
            Pick a random bacteria $x_k$ within **sensing_radius** of $x_j$
            Update orientation: $d_j = (x_k - x_j) / |x_k - x_j|$
        end
    end
    x=x+d***speed***dt
end
```



- Movies: switching rate = 3   Switching rate = 2   Switching rate = 0.5

One-dimensional ODE lattice model (après Galante-Levy)

- At each time-step:
  - Bacteria either moves to an adjacent cell according to its current orientation with rate a
  - Switches orientation with rate c
- ODE lattice model on n bins:

$$\frac{dU_j}{dt} = a(U_{j-1} + U_{j+1}) - (2a + c)U_j + c\left(U_{j-1}\eta^{+}_{j-1} + U_{j+1}\eta^{-}_{j+1}\right) \tag{1}$$

where $$\eta^{\pm}_j = \frac{\sum_{k=1}^{d} U_{j\pm k}}{\sum_{k=1}^{d}\left(U_{j+k} + U_{j-k}\right)}. \tag{2}$$

Stability of the homogeneous state

The model

$$\frac{dU_j}{dt} = a(U_{j-1} + U_{j+1}) - (2a + c)U_j + c\left(U_{j-1}\eta^{+}_{j-1} + U_{j+1}\eta^{-}_{j+1}\right)$$

has admits a homogeneous state $U_j = U$. Linearize around it:

$$U_j(t) = U + \phi_j e^{\lambda t}$$

Then

$$\lambda\phi_j = \left(a + \frac{c}{2} + \frac{c}{4d}\right)(\phi_{j-1} + \phi_{j+1}) - \left(2a + c - \frac{c}{2d}\right)\phi_j$$
$$- \frac{c}{4d}\left(\phi_{j+d} + \phi_{j-d} + \phi_{j+d+1} + \phi_{j-d-1}\right)$$

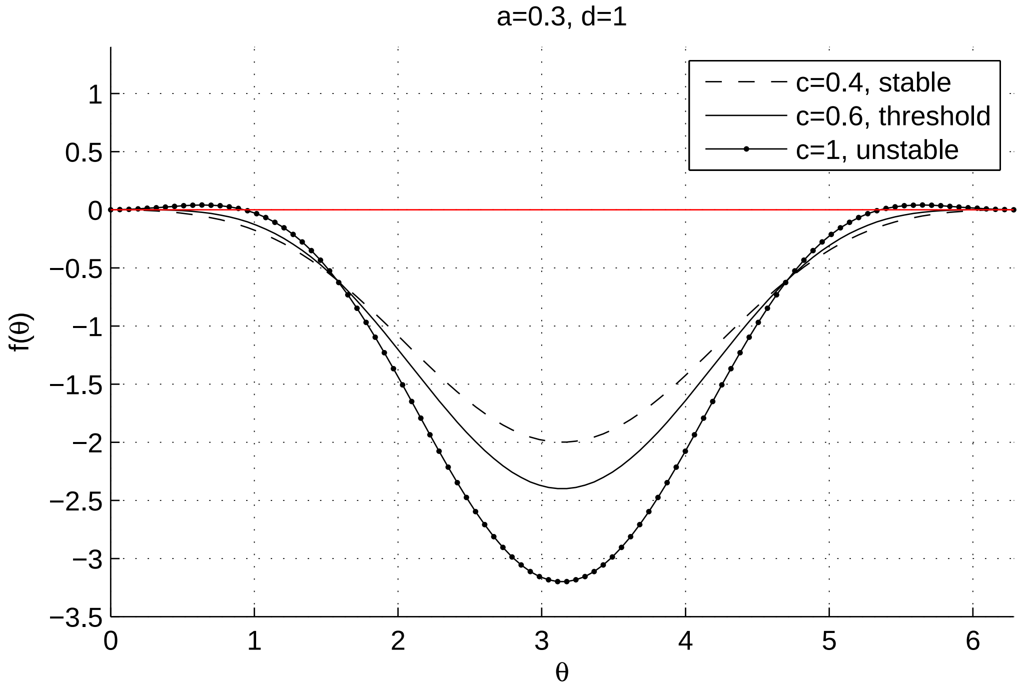
Anzatz:

$$\phi_j = \phi e^{\lambda t} e^{\frac{2\pi m j i}{n}}; \quad m = 0 \ldots n-1$$

$$\boxed{\lambda = f\left(\tfrac{2\pi m}{n}\right), \quad m = 0 \ldots n-1,}$$

$$\boxed{f(\theta) = (2a + c)\left(\cos(\theta) - 1\right) + \tfrac{c}{2d}\left(1 + \cos(\theta) - \cos(d\theta) - \cos((d+1)\theta)\right)}$$

$$f(\theta) = (2a + c)\left(\cos(\theta) - 1\right) + \frac{c}{2d}\left(1 + \cos(\theta) - \cos(d\theta) - \cos((d+1)\theta)\right)$$



- $f(0) = 0, \;\; f'(0) = 0,$
- Can be shown that $f(\theta) \leq 0$ for all θ iff $f''(0) = -2a + dc < 0,$
- **Homogeneous state is stable if** $c < c_0$; **unstable if** $c > c_0$, **where**

$$c_0 = 2a/d.$$

Continuum limit

$$\frac{dU_j}{dt} = a(U_{j-1} + U_{j+1}) - (2a + c)U_j + c\left(U_{j-1}\eta^+_{j-1} + U_{j+1}\eta^-_{j+1}\right)$$

- Let $U_j(t) = u(x,t)$ so that $U_{j+k}(t) = u(x + kh, t)$
- Expanding up to $O(h^2)$, we get:

$$u_t = -h^2\left(c\frac{d}{2} - a\right)u_{xx}$$

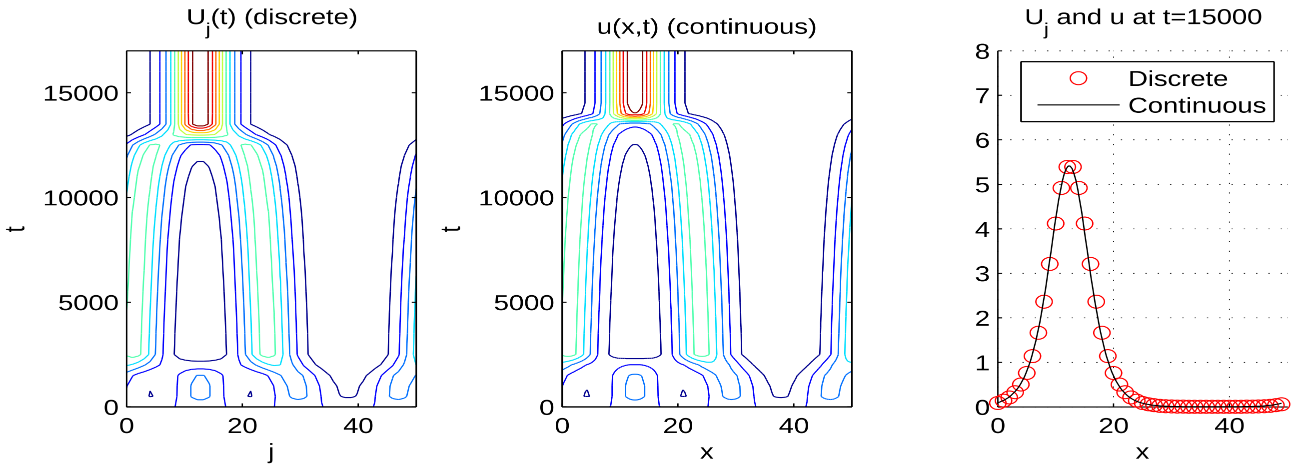
  - This recovers the linear stability threshold $c\frac{d}{2} - a = 0$.

- Expanding to $O(h^4)$ we get:

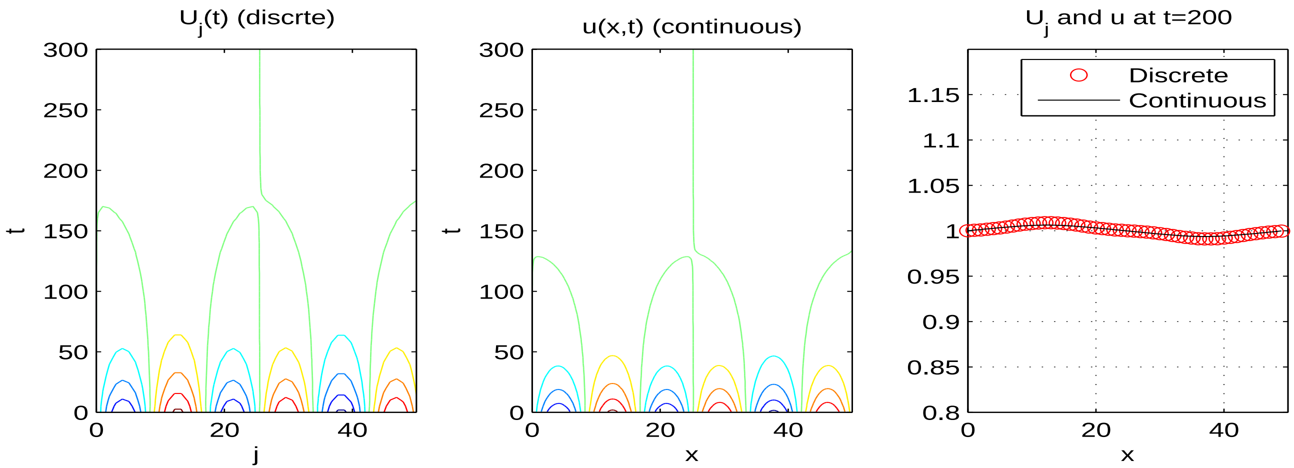
$$u_t = -Au_{xx} - Bu_{xxxx} + C\left(\frac{u_x u_{xx}}{u}\right)_x, \quad v_t = (a + c)(u - v);$$

$$A = h^2\left(c\frac{d}{2} - a\right); \quad B = \frac{h^4}{12}\left(\frac{c}{2}\left[1 + d\left(d^2 + 2d + 3\right)\right] - a\right);$$

$$C = \frac{h^4}{24}c(2d + 1)(d + 1)^2.$$



$$n = 50, a = 1, c = 2.2, d = 1.$$



$$n = 50, a = 1, c = 1.8, d = 1.$$

- Suppose that $c\frac{d}{2} - a > 0$ (i.e. homogeneous state unstable). Scale $x = \hat{x}\,(B/A)^{1/2}\,; \quad t = \hat{t}BA^{-2}$. After dropping the hats we then obtain

$$\boxed{u_t = -u_{xx} - u_{xxxx} + \alpha\left(\frac{u_x u_{xx}}{u}\right)_x} \tag{3}$$

where

$$\alpha := \frac{c\,(2d+1)\,(d+1)^2}{(c\,[1+d\,(d^2+2d+3)] - 2a)}. \tag{4}$$

Inhomogeneous steady state

$$u_t = -u_{xx} - u_{xxxx} + \alpha \left(\frac{u_x u_{xx}}{u}\right)_x$$

Set $u_t = 0$ and assume u decays at ∞ :

$$0 = -u_x - u_{xxx} + \alpha \frac{u_x u_{xx}}{u}. \tag{5a}$$

Because of scaling symmetry $u \to \lambda u$, we get a reduction of order:

$$u = \exp(v); \quad v_x = z$$

$$z'' + z + (3-\alpha) z z' + (1-\alpha) z^3, \quad z = u_x/u.$$

Write it as:

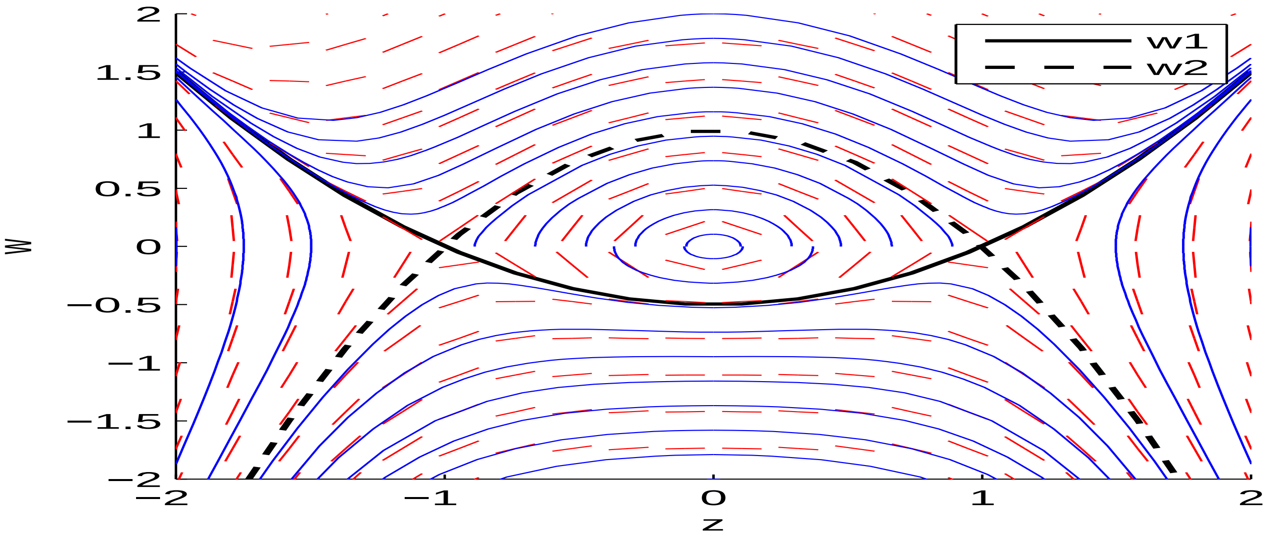
$$\frac{dz}{dx} = w; \tag{6}$$

$$\frac{dw}{dx} = -z + (\alpha - 3) z w + (1-\alpha) z^3. \tag{7}$$

Get 1st order Abel ODE:

$$\frac{dw}{dz} = \frac{-z}{w} - (3-\alpha) z - (1-\alpha) \frac{z^3}{w}. \tag{8}$$

Phase portarat:



The two saddles are connected by heteroclinic orbits of the form of a parabola. So try **Anzatz:**

$$w = Az^2 + B.$$

There are two solutions:

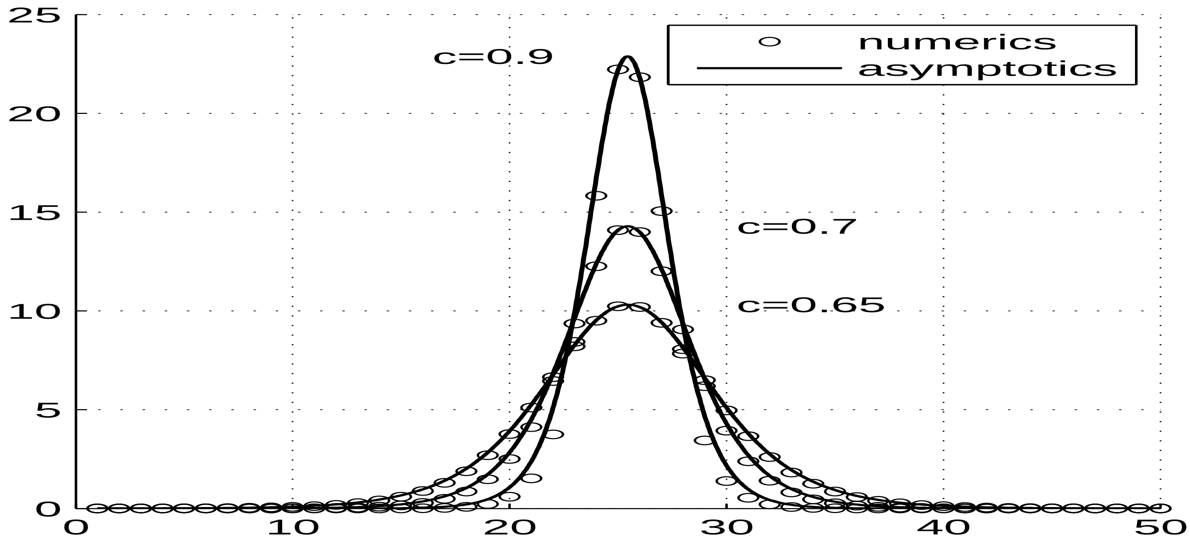
$$w_1 = \frac{(\alpha - 1)z^2 - 1}{2}; \quad w_2 = -z^2 + \frac{1}{\alpha - 1}.$$

Substitute $w = w_1$ into (5) yields

$$\frac{dz}{dx} = \frac{(\alpha - 1)z^2 - 1}{2} \tag{9}$$

Solve it and unwind the transformations to get

$$\boxed{u(x) = C\left[\,\text{sech}\left(\frac{\sqrt{\alpha-1}}{2}x\right)\right]^{\frac{2}{\alpha-1}}.} \tag{10}$$



Stability

- Linearized problem:

$$\lambda\phi = -\phi_{xx} - \phi_{xxxx} + \alpha\frac{\partial}{\partial x}\left(\frac{u_0'}{u_0}\phi_{xx} + \frac{u_0''}{u_0}\phi_x - \frac{u_0''u_0'}{u_0^2}\phi\right)$$

$$\text{where } u_0(x) = \left[\text{sech}\left(\frac{\sqrt{\alpha-1}}{2}x\right)\right]^{\frac{2}{\alpha-1}}.$$

- Exponentially small eigenvalue appears stable (resposible for metastability)

- Numerics suggest there is a continuum spectrum along negative real axis, no other real eigenvalues.

- Open problem: Prove stability!

Conclusions

- We derived a novel PDE model of bacterial aggregation: $\boxed{u_t = -u_{xx} - u_{xxxx} + \alpha \left(\frac{u_x u_{xx}}{u}\right)_x}$
- Explicit spike profile: $\boxed{u(x) = C \left[\text{sech} \left(\frac{\sqrt{\alpha - 1}}{2} x \right) \right]^{\frac{2}{\alpha - 1}}.}$
- Open questions:
  - Structural stability?
  - Metastability?
- To appear, Nonlinearity. Preprint: `http://www.mathstat.dal.ca/~tkolokol/bacteria.pdf`

Thank you! Any questions?