# MATH 5230/4230, Homework 1

Due: Thurs, Jan 24

1. The Van der Pol oscillator is governed by the ODE

$$u_{tt} + \varepsilon u_t \left(u^2 - 1\right) + u = 0. \tag{1}$$

   (a) Using the method of multiple scales, show that the leading order expansion is of the form

$$u = A(\varepsilon t) \cos\left(t + \Phi\left(\varepsilon t\right)\right)$$

   and find the equations for $A$ and $\Phi$.

   (b) Solve for $A$ and $\Phi$ subject to initial conditions $u\left(0\right) = 1, \quad u_t\left(0\right) = 0.$

   (c) Use a computer to plot a numerical solution to (1) for $\varepsilon = 0.1,\ t \in (0, 100)$. On the same graph, plot the solution you found in part b and the envelope $A\left(\varepsilon t\right)$. Comment on what you observe. Note: If using maple, see sample code on the course webpage.

2. Consider the system

$$\begin{cases} x' = -x + ay + x^2, \\ y' = -2x + ay. \end{cases}$$

(a) Show that zero equilibrium undergoes a Hopf bifurcation when $a = 1$.

(b) Eliminate $y, y'$ to show that $x(t)$ satisfies a second order ODE

$$x'' = -ax + x'\left(a - 1\right) + 2xx' - ax^2. \tag{2}$$

(c) Rescale $x = \varepsilon u(t)$ and let $a = 1 + \varepsilon^p$, where $p$ is chosen such that $u(t) = O(1)$. Show that to obtain a bounded solution, one choose have $p = 2$.

(d) Use the method of multiple scales (or Lindstead method) on to determine the amplitude of $u(t)$. Comment on what this says about the behaviour of the original system near the Hopf bifurcation.

3. A model of a laser subject to opto-electronic feedback is desribed by the following system [Erneux]:

$$x' = -y - \eta(1 + y(s - \theta)); \qquad y' = (1 + y)x. \tag{3}$$

The parameters $\eta$ and $\theta$ represent feedback parameter and delay time, respectively.

   (a) Linearize around the steady state $x = 0$, $y = -\eta/(1 + \eta)$. What is the transcendental equation for the resulting eigenvalue $\lambda$?

   (b) Seek Hopf bifurcations, i.e. plug in $\lambda = i\omega$, then separate real and imaginary parts. Show that Hopf bifurcations occur when

$$\sin\left(\omega\theta\right) = 0 \quad \text{or} \quad \left(1 + \eta\right)\omega^2 = 1 + \eta \cos\left(\omega\theta\right). \tag{4}$$

   (c) Equations (4) represent curves in the $(\eta, \theta)$ plane. Plot these curves. Analytically classify any intersection points that you may observe. The first such point occurs when $\theta = 2\pi$, $\eta = 3/5$. What are the possible values of $\omega$ at this point? List at least three more such intersections.

   (d) [GRAD STUDENTS ONLY] These intersection points are called double-hopf points and very interesting dynamics can be observed near these points. Perform numerical experiments of (3) with $\theta$ near $2\pi$, and with $\eta$ near $3/5$. Play around with $\eta$ and $\theta$ near these values and describe what you observe. Hand in any useful plots of numerics. REMARK: You can use matlab for solving delay ode's, see `dde23` command. Or you can use the free "Dynamics Solver" program (google it). If in doubt, ask me.