MATH 5230/4230, Homework 2

Due: Thurs, Oct. 9

1. The Van der Pol oscillator is governed by the ODE
$$u_{tt} + \varepsilon u_t \left(u^2 - 1\right) + u = 0. \tag{1}$$
   (a) Using the method of multiple scales, show that the leading order expansion is of the form
$$u = A(\varepsilon t) \cos\left(t + \Phi\left(\varepsilon t\right)\right)$$
   and find the equations for A and Φ.

   (b) Solve for A and Φ subject to initial conditions $u\left(0\right) = 1, \quad u_t\left(0\right) = 0.$

   (c) Use a computer to plot a numerical solution to (1) for $\varepsilon = 0.1,\ t \in (0, 100)$. On the same graph, plot the solution you found in part b and the envelope $A\left(\varepsilon t\right)$. Comment on what you observe. Note: If using maple, see sample code on the course webpage.

2. Consider the ODE
$$u_{tt} + u = \varepsilon u^2; \qquad u(0) = 2; \quad u'(0) = 0. \tag{2}$$
   (a) Use Maple to solve numerically (2) with $\varepsilon = 0.1$ and plot the solution for $t = 0 \ldots 200$, superimposed with $2\cos(t)$. Comment on the "phase drift" that you observe. Hand in the printout. Here is some useful code for ya:

```
restart; eps := 0.1;
sol := dsolve({diff(u(t),t,t)+u(t)=eps*u(t)^2, u(0)=2,D(u)(0)=0},numeric);
with(plots):
odeplot(pp, [[t,u(t)],[t,2*cos(t)]], 0..200,numpoints=2000);
```

   (b) Apply the method of Multiple scales to (2). HINTS:

   - You will need to expand to THREE orders, i.e. write $u(t) = U(t, s, \tau) = U_0(t, s, \tau) + \varepsilon U_1(t, s, \tau) + \varepsilon^2 U_2(t, s, \tau) + \ldots$ where $s = \varepsilon t$, $\tau = \varepsilon^2 t$.
   - You will find that $U_0(t, s, \tau) = U_0(t, \tau) = A(\tau)e^{it} + \overline{A(\tau)}e^{-it}$ by eliminating secular terms at $O(\varepsilon)$ order. Then solve explicitly for U_1.
   - Finally, determine the correction to the period by eliminating secular terms at $O(\varepsilon^2)$ order

   (c) In part (b), you should have found that $u(t) \sim 2\cos(t\left(1 + \varepsilon^2\omega\right))$ for some ω that you had to determine. What is that ω?

   (d) Modify the code in part (a) to compare full numerical solution with $2\cos(t\left(1 + \varepsilon^2\omega\right))$. Contrast the resulting plot with the plot from part (a). Hand in the appropriate plots.

3. Consider the system
$$\begin{cases} x' = -x + ay + x^2, \\ y' = -2x + ay. \end{cases}$$
   (a) Show that zero equilibrium undergoes a Hopf bifurcation when $a = 1$.

   (b) Eliminate y, y' to show that $x(t)$ satisfies a second order ODE
$$x'' = -ax + x'\left(a - 1\right) + 2xx' - ax^2. \tag{3}$$
   (c) Rescale $x = \varepsilon u(t)$ and let $a = 1 + \varepsilon^2$ to obtain
$$u_{tt} + u = \varepsilon\left(-u^2 + 2uu_t\right) + \varepsilon^2\left(u_t - u\right) + O\left(\varepsilon^3\right). \tag{4}$$
   (d) Apply the method of multiple scales to (4) to determine the amplitude of $u(t)$ as $t \to \infty$. Comment on what this says about the behaviour of the original system near the Hopf bifurcation.

   (e) Verify your solution in (d) by using Maple to compare the result you obtained in (d) to the full numerical solution of (4). Hand in the appropriate printouts.

4. A model of a laser subject to opto-electronic feedback is desribed by the following system [Erneux]:

$$x' = -y - \eta(1 + y(s - \theta)); \qquad y' = (1 + y)x. \tag{5}$$

The parameters η and θ represent feedback parameter and delay time, respectively.

(a) Linearize around the steady state $x = 0$, $y = -\eta/(1+\eta)$. What is the transcendental equation for the resulting eigenvalue λ?

(b) Seek Hopf bifurcations, i.e. plug in $\lambda = i\omega$, then separate real and imaginary parts. Show that Hopf bifurcations occur when

$$\sin(\omega\theta) = 0 \quad \text{or} \quad (1+\eta)\,\omega^2 = 1 + \eta\cos(\omega\theta)\,. \tag{6}$$

(c) Equations (6) represent curves in the (η, θ) plane. Plot these curves. Analytically classify any intersection points that you may observe. The first such point occurs when $\theta = 2\pi$, $\eta = 3/5$. What are the possible values of ω at this point? List at least three more such intersections.

(d) [BUNUS MARKS] These intersection points are called double-hopf points and very interesting dynamics can be observed near these points. Perform numerical experiments of (5) with θ near 2π, and with η near $3/5$. Play around with η and θ near these values and describe what you observe. Hand in any useful plots of numerics. REMARK: You can use matlab for solving delay ode's, see `dde23` command. Or you can use the free "Dynamics Solver" program (google it). If in doubt, ask me.