

MATH 5230/4230, Homework 3

1. Consider the following problem:

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + \lambda e^u = 0 \\ u'(0) = 0, \quad u(1) = 0, \quad u > 0 \text{ inside } [0, 1) \end{cases} \quad (1)$$

- (a) Change the variables $v = e^u$.
- (b) Reformulate the boundary value problem (1) as an initial value problem for v .
- (c) Solve the problem in (b) numerically using a computer, and draw the bifurcation diagram of $u(0)$ vs. λ for $n = 1, 2, 3$. See website for Maple code to do this for MEMS problem. You should observe that (1) has infinitely many solutions if $n = 3$ and $\lambda = 2$ but at most two solutions if $n = 1, 2$. Also observe that there exists λ^* such that solution to (1) exists if and only if $\lambda \leq \lambda^*$.
- (d) Prove the existence of λ^* and give an upper bound for it.
- (e) Use an appropriate change of variables and a phase-plane/stability analysis to study the bifurcation diagrams you sketched in (c).
- (f) The bifurcation diagram changes qualitatively again if $n > n^* > 3$. Determine this n^* and draw the bifurcation diagram for $n > n^*$.
- (g) Using the method of sub/supersolutions, give a lower bound for λ^* (see also question 3).
- (h) In one dimension $n = 1$, this problem can be solved explicitly. Do it and explicitly compute λ^* for $n = 1$. Does it agree with upper/lower bounds you found in parts (b) and (g)?

2. Consider the eigenvalue problem

$$-\Delta\phi = \lambda\phi \text{ inside } \Omega, \quad \phi = 0 \text{ on } \partial\Omega. \quad (2)$$

- (a) Show (as we did in class) that the principal eigenvalue satisfies

$$\lambda = \min_{\phi \in K} R(\phi), \quad \text{where } R(\phi) = \frac{\int_{\Omega} |\nabla\phi|^2}{\int_{\Omega} \phi^2}. \quad (3)$$

Here, K is the class of smooth functions with Dirichlet boundary conditions:

$$K = \{\phi : \phi \in C^\infty, \quad \phi = 0 \text{ on } \partial\Omega\}. \quad (4)$$

- (b) Use part (a) to show that increasing Ω decreases λ .
- (c) Show that the second eigenvalue solves (3) with K replaced by

$$K = \left\{ \phi : \phi \in C^\infty, \quad \phi = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} \phi\phi_1 = 0 \right\}. \quad (5)$$

where ϕ_1 is the principal eigenfunction of the problem (2).

3. In class, we encountered the following problem. Let μ be the principal eigenvalue to

$$\begin{aligned} -\Delta\phi &= \mu\phi \text{ inside } \Omega' \\ \phi &= 0 \text{ on } \partial\Omega' \end{aligned}$$

Given a constant a , we need to choose Ω' with $\Omega \subset \Omega'$ such that

$$\frac{\min_{\Omega} \phi}{\max_{\Omega'} \phi} \geq a.$$

- (a) Suppose that $\Omega = [-L, L] \subset \mathbb{R}$. How should you choose Ω' and what would the corresponding μ be?
- (b) Suppose that $\Omega \subset \mathbb{R}^2$. How should you choose Ω' and what would the corresponding μ be? Hint: use question 2 to replace the domain by an appropriate square domain...