MATH 5230/4230, Homework 3

Due: Thurs 21 Feb.

1. Consider the problem
$$\begin{cases} u_t = u_{xx} + 2u - 2u^3, & x \in (-L, L) \\ u_x(\pm L, t) = 0 \end{cases}$$
Assume that $L \gg 1$ and we look for the travelling wave solution of the form
$$u(x) \sim U_0(x - l(t))$$
where $U_0(z) = \tanh(z)$. and where $|l| \ll L$.
   (a) Derive the ODE for the position of the interface $l(t)$.
   (b) Using FlexPDE, verify the formula you obtained in part (a). Run the code with several appropriately chosen parameters and plot both the asymptotics and full numerics on the same graph. See sample code on the webpage.
   (c) Show that $l = 0$ is an unstable equilibrium to the ODE you derived in part (a). Verify that the eigenvalue of the $l = 0$ equilibrium is the same as the eigenvalue $\lambda = 196\exp(-4L)$ that we obtained in class when we computed that stability of the interface at the origin.
   (d) Let $l_0 = l(0)$ and suppose that $1 \ll l_0 \ll L$. In this case, simplify the ODE from part (b) using the identity $2\sinh(z) \sim e^z$ (for large z). Solve the resulting (simplified) ODE and determine (asymptotically) the time $T = T(l_0)$ at which the interface collides with the boundary. Verify this numerically with FlexPDE for some well-chosen parameters. How big is the relative error?

2. In this problem you are asked to construct radially propagating interface in 2D to the PDE
$$u_t = \varepsilon^2 \Delta u - 2(u + \varepsilon a)(u - 1)(u + 1), \quad x \in \mathbb{R}^2.$$
where we assume that $0 < \varepsilon \ll 1$ and $a = O(1)$; we allow a to be either positive or negative.
   (a) Find a travelling wave solution of the form
$$u(x) \sim U_0\left(\frac{r - r_0(\varepsilon^2 t)}{\varepsilon}\right), \; r = |x|,$$
where $U_0(z) = \pm\tanh(z)$. Derive an ODE for r_0; note that you get two different answers depending on whether you take $+\tanh$ or $-\tanh$. HINT: $\Delta u = u_{rr} + \frac{1}{r}u_r$.
   (b) Under what conditions does the ODEs you derived in part (a) have an equilibrium point? Is that equilibrium stable or unstable?

3. Suppose that $\Omega \subset \mathbb{R}^3$ is a *thin tube,* defined as
$$\Omega = \{(x, y, z) : (y, z) \in \delta D(x), \quad x \in (a, b)\}$$
where $\delta \ll 1$ and $D(x) \subset \mathbb{R}^2$ is the cross-section of the tube at x that varies smoothly with x. Show that in the limit $\delta \ll 1$, the system
$$\begin{cases} u_t = \Delta u + f(u), & (x, y, z) \in \Omega \\ \partial_n u = 0 \text{ on } \partial\Omega \end{cases}$$
reduces to
$$\begin{cases} u_t = u_{xx} + \frac{A'(x)}{A(x)} u_x + f(u), & x \in (a, b) \\ u_x = 0 \text{ at } x = a, b. \end{cases}$$
where $A(x)$ is the area of $D(x)$. Hint: make a change of variables $y = \delta\bar{y}, \quad z = \delta\bar{z}, x = x$.