MATH 5230/4230, Homework 5

Due: Thurs 21 March.

1.

   (a) Solve the PDE $u_t + xu_x = 0$ subject to initial condition $u(x,0) = x$.

   (b) Solve the PDE $u_t + xu_x = u$ subject to initial condition $u(x,0) = x + 1$.

2. Consider the system

$$u_t + (2-u)u_x = 0; \qquad u(x,0) = 1 + \tanh(x).$$

   (a) Determine the characteristic curves for this ODE.

   (b) Show that the solution develops a shock and compute the time $t = t_s$ at which the shock first occurs.

   (c) Sketch the solution profile $u(x,t)$ for $t = 0, \quad 0.5t_s, \quad t_s, \ 1.5t_s$.

3.

   (a) Consider the system

$$u_t + uu_x = 0; \quad u(x,0) = \phi(x).$$

   where ϕ is is given by

$$\phi(x) = \begin{cases} 1, & x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 0. \end{cases} \tag{1}$$

   The initial condition develops a shock at some time $t = t_s$. Compute the value of t_s and sketch the solution at $t = 0, t = t_s/2$ and $t = t_s$.

   (b) Repeat part (a) but with

$$u_t + uu_x = u; \quad u(x,0) = \phi(x).$$

4. Consider the system

$$u_t + uu_x - u = \varepsilon u_{xx}; \quad u(x,0) = \phi(x).$$

with initial condition given by

$$\phi = \begin{cases} 1, x < 0 \\ 0, x > 0 \end{cases}.$$

   (a) This system forms a shock at $t = 0$, which is smoothed out by the diffusion term for $t > 0$. Compute the evolution equation for the location of the shock $x = s(t)$.

   (b) Repeat part (a) but with uu_x replaced by u^2u_x.

   (c) Use FlexPDE to verify what you found in parts (a) and (b) numerically. That is, you should hand in plots that compare the location of the interface, $s(t)$ computed numerically, with your asymptotic predictions. See website for a sample for Burger equation.

5.

   (a) Consider the shallow-water equations

$$u_t + uu_x + 2cc_x = 0, \qquad 2c_t + 2uc_x + cu_x = 0$$

   where $c = \sqrt{gh}$ with u, h as in class. Using the method of charcteristics, solve the *dam-breaking problem* which corresponds to the initial conditions $u = 0$ and $c = \begin{cases} c_0, & x \le 0 \\ 0, & x \ge 0 \end{cases}$ at $t = 0$. HINT: consider the characteristics along $x = 0, t = 0$ with $u = 0$ and $c = 0 \ldots c_0$.

   (b) Set $c_0 = 1$ and sketch the solution $h(x)$ at time $t = 0$,and $t = 1$.