Numerical computation of boundary value problems with very sharp layers

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Introduction

- Singular perturbation problems depend on a small parameter ε which typically premultiplies the highest derivative.
- As $\varepsilon \to 0$, the problems exhibit localized structures such as boundary layers, corner layers, spikes, interfaces.
- Typically, the localized structure has the size $O(\varepsilon)$; the solution is relatively smooth outside the localized region.
- Standard codes to solve BVP have difficulty resolving localized structures: typically, meshsize scales with $1/\varepsilon$.
- Example: a standard code requires 10,000 meshpoints when $\varepsilon = 10^{-5}$?.

Problem 1

Consider the problem

$$\varepsilon^2 u'' - u + \left(1 + x^2\right) u^2 = 0; \quad u'\left(0\right) = 0; \;\; \varepsilon u'(1) = -u(1).$$

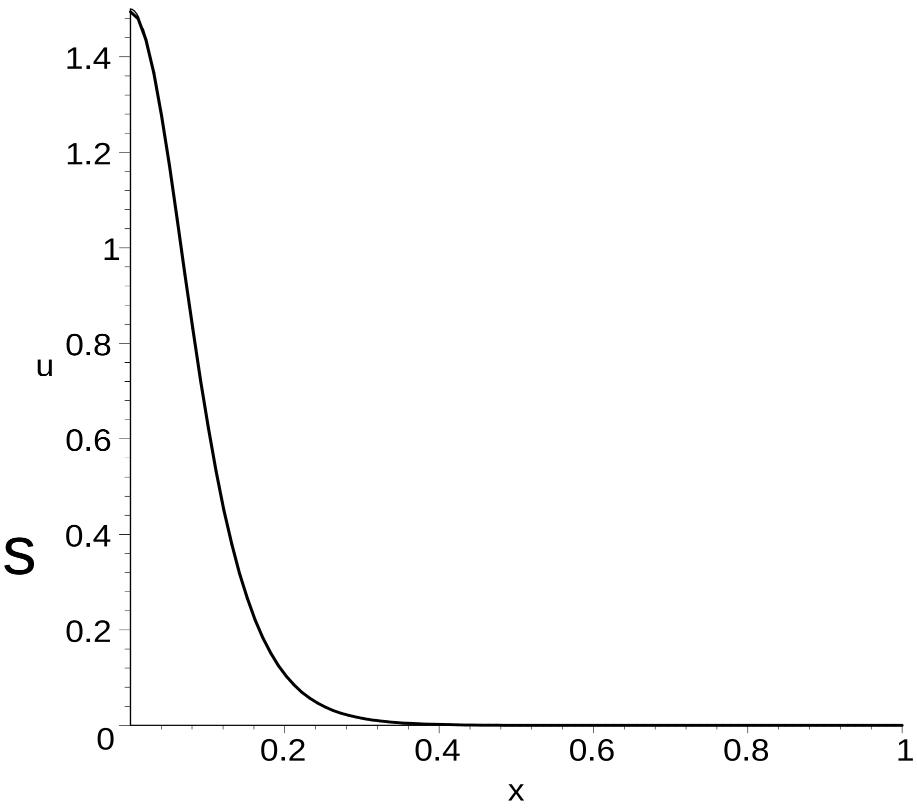
Asymptotic solution:

$$u(x) \sim w(\frac{x}{\varepsilon})$$

where

$$w(y) = \frac{3}{2}\,\mathrm{sech}^2\left(y/2\right)$$ solves

$$w_{yy} - w + w^2 = 0.$$



- Note that w decays for large y,

$$w(y) \sim 6e^{-y} \quad \text{for large} \quad y.$$

- This exponential decay causes trouble for BVP solvers.
- The solution exhibits two different spatial scales.
- Standard BVP solver: meshsize scales like $1/\varepsilon$.

Split-domain method

- Choose $l \in [0,1]$,
$$\varepsilon \ll l \ll 1$$
.
- On $[0,l]$, (inner problem) transform:
$$x = ly, \;\; u(t) = \hat{u}(y)$$
- On $[l,1]$, (outer problem) transform:
$$x = l + (1-l)y, \;\; u(t) = \exp\left(\frac{\tilde{u}(y)}{\varepsilon}\right)$$

- We get a 4-dimensional BVP for $\hat{u}, \tilde{u}$ on $y \in [0,1]$. Two addional constraints impose continuity of u and u' at l:

$$\hat{u}'(0) = 0, \quad \tilde{u}'(1) = -1$$

$$\hat{u}(1) = \exp\left(\frac{\tilde{u}(0)}{\varepsilon}\right) \quad \text{(continuity of } u\text{)}$$

$$\frac{\hat{u}'(1)}{l} = \frac{1}{\varepsilon(1-l)} \exp\left(\frac{\tilde{u}(0)}{\varepsilon}\right) \tilde{u}'(0) \quad \text{(continuity of } u'\text{)}$$

- The parameter l is chosen by trial and error. Global tolerance is set to 10^{-6}; Maple's dsolve/numeric collocation code is used with adaptive gridding.

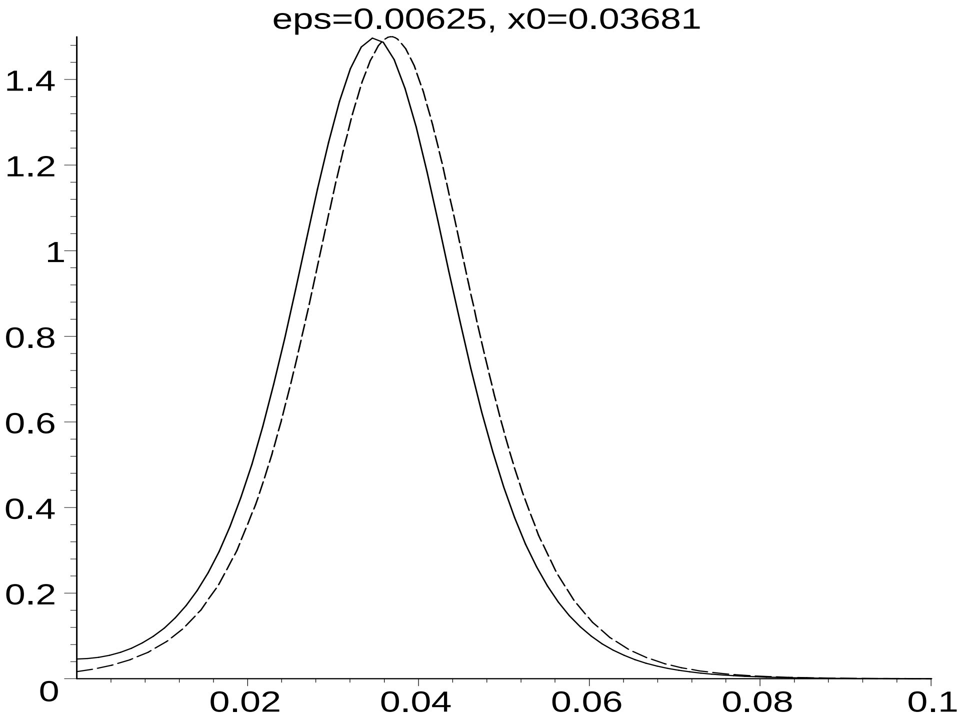
Meshsize scaling laws

ε	standard	$l = 9\varepsilon$	$l = 4\varepsilon \ln \frac{1}{\varepsilon}$
0.1	51	21	21
0.05	87	21	24
0.025	106	21	21
$2^{-3} \times 0.1$	178	21	26
$2^{-4} \times 0.1$	376	21	29
$2^{-5} \times 0.1$	792	30	30
$2^{-6} \times 0.1$	error	58	32
$2^{-7} \times 0.1$		119	31
$2^{-8} \times 0.1$		226	32
$2^{-9} \times 0.1$		472	33
$2^{-10} \times 0.1$		946	34
$2^{-11} \times 0.1$		error	35
$\ldots$			...
$2^{-16} \times 0.1$			41
$2^{-17} \times 0.1$			42
$2^{-18} \times 0.1$			error

Problem 1b

Same as Problem 1, but it has *another* solution of the form $u \sim w\left(\frac{x-x_0}{\varepsilon}\right)$ where y_0 satisfies:

$$x_0 = \varepsilon \frac{1}{2} \ln \left(\frac{30}{\varepsilon x_0} \right)$$



Challenge: Compute Probelm 1b with $\varepsilon = 10^{-4}$.

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Two different scales

- To leading order, $x_0 = \varepsilon\frac{1}{2}\ln\left(\frac{30}{\varepsilon}\right)$ has order $O(\varepsilon \ln \varepsilon)$
- On the other hand, spike extend is of $O(\varepsilon)$.
- The ratio of two scales is $O(1/\ln \varepsilon)$.
- This means that to compare asymptotics of Problem 2, we must take ε *exponentially* small!

Problem 2

Find the principal eigenvalue of:

$$\Delta\phi + \lambda\phi = 0 \text{ inside } B_1 \backslash B_\varepsilon$$
$$\phi = 0 \text{ on } \partial B_\varepsilon$$
$$\partial_n \phi = 0 \text{ on } \partial B_1$$

in the limit $\varepsilon \to 0$.
This is equivalent to an ODE BVP:

$$(r\phi_r)_r + \lambda r\phi = 0; \quad \phi(\varepsilon) = 0; \quad \phi'(1) = 0 \tag{1}$$

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0.1 Asymptotic solution

Define

$$\eta = \frac{1}{\ln \frac{1}{\varepsilon}}.$$

Note that

$$\varepsilon \ll \eta \ll 1.$$

Two-term asymptotic form of the eigenvalue:

$$\lambda_{asymptotic} \sim 2\eta + \frac{3}{2}\eta^2 \tag{2}$$

$$\phi \sim \lambda \left(\frac{1}{2} \ln(r/\varepsilon) - \frac{r^2}{4} \right) \tag{3}$$

Derivation of asymptotic solution: Assume $\lambda \ll 1$ and expand in λ :

$$\phi = 1 + \lambda\phi_1 + ...$$

so that

$$(r\phi_{1r})_r + r = 0;$$
$$1 + \lambda\phi_1(\varepsilon) = 0;$$
$$\phi_1'(1) = 0.$$

Then

$$\phi_1 \sim -\frac{1}{\lambda} + \frac{1}{2}\ln(\frac{r}{\varepsilon}) - \frac{r^2}{4}$$

Solvability condition:

$$\lambda \int_{\varepsilon}^{1} \phi r dr \sim \varepsilon \phi'(\varepsilon) \sim +\frac{1}{2}\lambda$$

$$\int_{\varepsilon}^{1} \phi_1 r dr \sim -\frac{1}{2\lambda} + \frac{1}{4} \ln \frac{1}{\varepsilon} - \frac{3}{16}$$

$$\lambda \sim \frac{2}{\ln \frac{1}{\varepsilon} - \frac{3}{4}}$$

Exact solution given implicitly by:

$$\phi = J_0(\sqrt{\lambda} r) - \frac{J_0'(\sqrt{\lambda})}{Y_0'(\sqrt{\lambda})} Y_0(\sqrt{\lambda} r);$$

$$J_0(\sqrt{\lambda}\varepsilon)Y_0'(\sqrt{\lambda}) - J_0'(\sqrt{\lambda})Y_0(\sqrt{\lambda}\varepsilon) = 0.$$

Numerical solution, standard formulation

- Solve the “augmented system”,

$$(r\phi_r)_r + \lambda r\phi = 0; \quad \phi(\varepsilon) = 0; \quad \phi'(1) = 0;$$
$$\lambda_r = 0; \quad \phi(1) = 1.$$

  Use $\lambda_i = 0, \phi_i = \ln(r/\varepsilon)$ as initial guess; solve stating with $\varepsilon = 0.1$ and use continuation.

- Mesh size grows like $O(\frac{1}{\varepsilon})$; the eigenvalue is of $O\left(1/\ln\varepsilon\right)$. **Reason:** the solution has a log singularity near $x \sim \varepsilon$ (looks like $\ln\frac{r}{\varepsilon}$).
- Adaptive mesh doesnt seem to help (at least not using Maple’s dsolve)

Numerical solution, transformed formulation

- Change variables

$$t = \ln r; \qquad \phi(r) = \Phi(t);$$

$$e^{-2t}\Phi''(t) + \lambda\Phi = 0; \quad \Phi\left(\ln \varepsilon\right) = 0; \ \ \Phi'(0) = 0 \qquad (4)$$

- The resulting problem solved with standard code
- Global error tolerance of 10^-6 is used.

Comparison of meshsize

ε	standard/fixed	standard/adaptive	Transformed
0.05	76	64	
0.01	407	120	19
0.005	880	274	
0.0025	1903	573	
10^{-3}		1623	18
10^{-4}			19
10^{-5}			21
10^{-6}			25
10^{-7}			29
10^{-8}			30
10^{-9}			33
10^{-10}			36

- For $\varepsilon = 10^{-10}$ we get

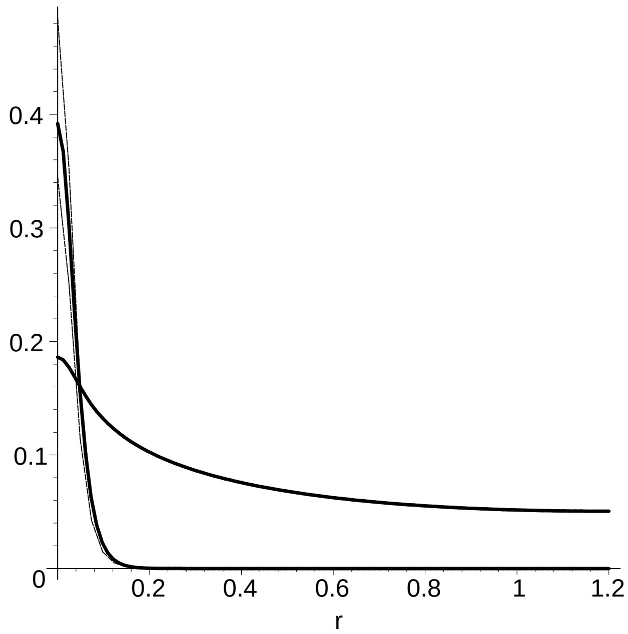
$$\begin{aligned} \varepsilon = 10^{-10}; \quad \eta &= 0.0434294 \\ \lambda_{\text{numeric}} &= 0.089757 \\ \lambda_{\text{asymptotic,1}} &= \mathbf{0.08}6850 = 2\eta \\ \lambda_{\text{asymptotic,2}} &= \mathbf{0.0896}88 = 2\eta + \frac{3}{2}\eta^2 \end{aligned}$$

Conclusion: two-term expansion seems to be correct.

Problem 3: Gierer-Meinhardt system in 2D

$$\varepsilon^2 \left(u_{rr} + \frac{1}{r} u_r \right) - u + u^2/v = 0; \quad v_{rr} + \frac{1}{r} v_r - v + u^2 = 0; \quad r \in [0, L]$$

$$u'(0) = v'(0) = u'(L) = v'(L)$$



$\varepsilon = 0.025$; thin lines are one and two-order asymptotic approximation.

Asymptotic solution:

$$u \sim \xi w\left(\frac{r}{\varepsilon}\right);$$

$$v \sim \begin{cases} \xi, \ r \ll \varepsilon \\ \xi \frac{1}{2\pi}\left[K_0(r) - \frac{K_0'(L)}{I_0'(L)} I_0(r)\right], \quad r \gg \varepsilon \end{cases}$$

where

$$w_{yy} + \frac{1}{y} w_y - w + w^2 = 0 \text{ with } w'(0) = 0, \ \ w \to 0 \text{ as } y \to \infty$$

and

$$\xi \sim \xi_0 + \eta \xi_1 + \cdots; \quad \eta = \frac{1}{\ln \frac{1}{\varepsilon}};$$

$$\xi_0 = \frac{1}{\int_0^\infty w^2(s) s ds} = 0.20266265$$

- To 5 decimal places,

$$\xi_1 = (0.38330 - 2H_0)\,\xi_0$$

where

$$H_0 = \ln 2 - \gamma - \frac{K_0'(L)}{I_0'(L)}.$$

- Leading order asymptotics have $O(\frac{1}{\ln 1/\varepsilon})$ error
- If $\varepsilon = 0.025$ then $\eta = 0.27$, not very small!!
- To verify ξ_1 numerically, we need to solve this problem for "exponentially small" ε, say $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$.

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Numerical solution using standard formulation

- To handle the singularity at $r = 0$, write $u_{rr} + \frac{1}{r}u_r = f(u)$; then expand around $r = 0$, for small h the BC becomes:

$$u'(h) \sim \frac{1}{2}f(u(h))h + O(h^2)$$

- Choose $h = 10^{-6}$; $L = 1$;
- Using continuation and adaptive grid, we can get solution up to $\varepsilon = 10^{-3}$ ($\eta = 0.14476)$ but it requires 1500 meshpoints with $L = 1$, global error = 10^{-3}
- To better verify ξ_1 numerically, we would like to take $\eta = 0.1 \implies \varepsilon \sim 4.5 \times 10^{-5}$.

Numerical solution using split domain:

- Choose $h = 10^{-2}\varepsilon$, and shift the domain $r = h + (L-h)t$, $t = [0,1]$.
- Choose $l \in [0,1]$, $\varepsilon \ll l \ll 1$.
- On $[0,l]$, transform:

$$t = ly, \quad u(t) = \hat{u}(y)$$

- On $[l,1]$, transform:

$$t = l + (1-l)y, \quad u(t) = \exp\left(\frac{\tilde{u}(y)}{\varepsilon}\right)$$

- Using $l = 4\varepsilon \ln \frac{1}{\varepsilon}$...

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Comparison of mesh size

ε	$\eta = 1/\ln(1/\varepsilon)$	Standard	Split domain
0.01	0.217	60	44
0.005	0.189	132	102
10^{-3}	0.145	709	352
5×10^{-4}	0.132	≥ 2000	704
10^{-4}	0.11		≥ 2000

- Here, split domain is only a slight improvement!!!
- Challenge: Can you compute with $\varepsilon = 10^{-7}$?

Challenges

- Automate layer detection and domain splitting
- How to choose the optimal l numerically?
- What is the theoretical optimal scaling law for the mesh size, as a function of ε?
- How to find the optimal transformation numerically?
- Interior spikes?
- Challgenge: Compute Problem 1b with $\varepsilon = 10^{-5}$.
- Challgenge: Compute Problem 3 with $\varepsilon = 10^{-7}$.