Mathematics on the chopping block

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ABSTRACT.
The production of end-grain wooden cutting boards involves repeated operations that leads to interesting mathematical questions. I explore some of the mathematical issues that arise in the process, which pose some interesting mathematical puzzles. I also show how mathematics can be used to create intricate artistic designs for cutting boards that are amenable to woodworking. These designs have been tested in real wood. Some of them belong to a class of iterated function systems fractals but others do not. The overall goal is to produce an aesthetically pleasing design suggested by mathematics and implementable in wood.

1. INTRODUCTION

Suppose you want to make a wooden chess board. You have some dark wood – such as oak – and some light wood – such as birch – to make black and white squares. You also have a table saw and some wood glue. One possiblilty is to cut 32 white squares, 32 black squares, then glue them up. However arranging all the squares and trying to glue them all at once will inadvertanly introduce many imprefections. A much better way – and the way this is done in practice – is illustrated in Figure 1. First, cut eight strips of wood – four white and four black – then glue them together into a board, iterating black and white. Let the glue dry, then make seven cuts across resulting in 8 stripes as in figure 1. Flip every second stripe and re-glue. Voilà: a chess-board.

The above procedure requires two glueups and seven cuts after the initial eight strips are cut. Each glue-up is along one direction only, which facilitates the use of clamps to fix the glue while it dries. A similar procedure – using only two glue-ups is used to make surprisingly complex and artistic end-grain cutting boards, some of which are illustrated in Figure 2. These boards were produced by Andrey Muntian (MTMWood) who is the master of the craft [1]. It is a nice mathematical puzzle for the reader to try to deduce the process which generates these patterns (see Andrey's website referenced in the figure for answers).

In this paper we discuss several techniques for producing a large variety of cutting board designs. Some examples – constructed in wood – are illustrated in Figures 2, 3. We explore connections to various branches of mathematics, including group theory, combinatorics, and fractals. We also propose several open mathematical questions that are motivated by making various designs. On the other hand, some of the designs provide interesting challenges for woodworkers as well.

There is a fine balance between what an artist or computer can do, what is aestethically pleasing, and what is implementable in wood in a workshop without fancy tools such as a CNC router and within a reasonable time/budget. For the most part, I emphasize realistically obtainable designs, as tested by actually building them. The primary goal is to produce an artistic design using mathematical techniques as a guide, and to motivate the reader to try out

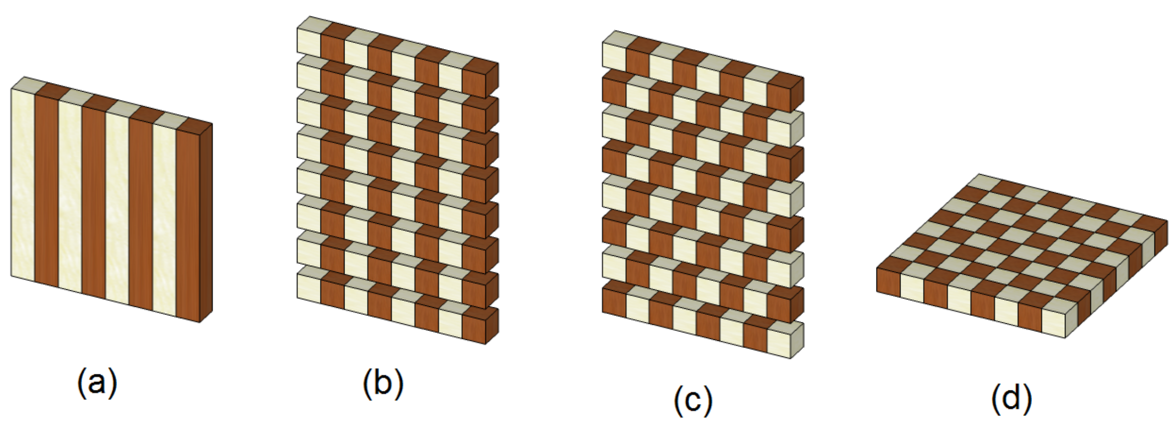


FIG. 1. Making an end-grain chessboard. (a) glue together alternating strips; (b) cut across; (c) flip every second strip (d) glue again, end-grain up.

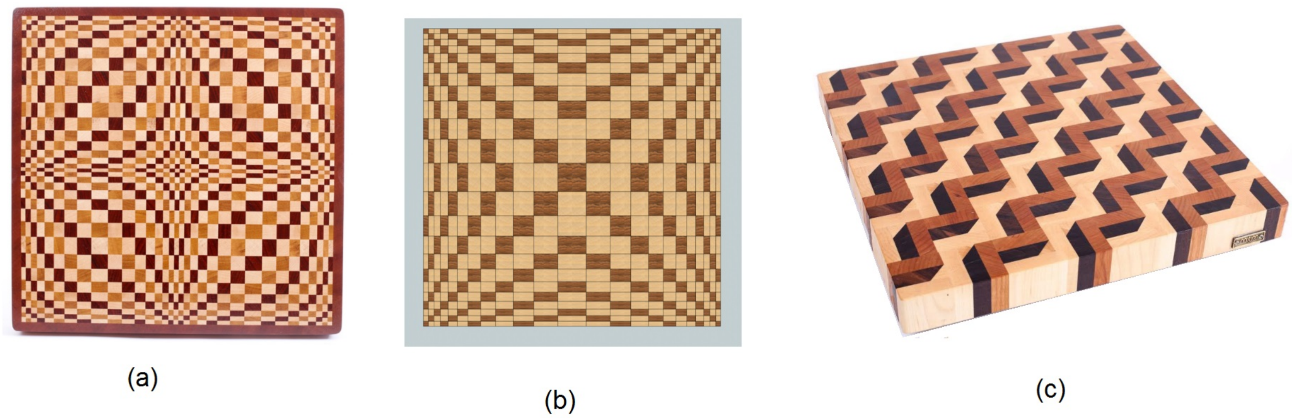


FIG. 2. Examples of artistic cutting boards, designed and built by Andrey Muntyan (MTMWood). Reproduced with Andrey's permission. (a) Multicolor Butterfly. (b) 3D board #4 (c) 3D board #10. Detailed videos and plans explaining how these boards were made are available on Andrey's website, mtmwood.com

some of these techniques to create and build their own designs.

2. BASIC SQUARE DESIGNS

As mentioned in the introduction, the basic technique is to glue some strips, cut them across, rearrange and reglue. After the second set of cuts, the wood should be turned end-grain up to create an end-grain cutting board. There are several reasons for doing this [1]. End-grain boards are best for of stability. They keep the knives relatively sharp since the knife actually goes through the wood fibers instead of across them. This also extends the lifespan of the board, since the wood fibres close up after each cut, rather than the knife cutting through the fibres. Moreover, the glue holds much better when applied along the fibres, and turning the wood end-grain up guarantees that all the glueing-up is done along the fibers.

How many different designs can be produced using two glue-ups? To formulate this question mathematically, assume we start with n black-and-white stripes. Glue them together horizontally, then make $m-1$ cuts to make m strips vertically, rearrange, and glue again. Let $N(n,m)$ be the number of resulting patterns.

Assuming two choices for each stripe (either black or white), there are 2^n possible stripe selections before the first glue-up, so that $N(n,1)=2^n$. After making m vertical strips, each strip can be either left alone or flipped vertically. Permuting resulting strips is allowed, but does not alter the pattern. This leads to following recursion relationship:

$$N(n,m+1) = 2\left(N(n,m) - x(n)\right) + x(n). \tag{2.1}$$

Here, $x(n)$ is the total possible number of n black-and-white squares in a column that remains invariant under flipping it. To count $x(n)$, suppose n is even. Then the first $n/2$ squares are chosen arbitrary, and the last $n/2$ squares are a reflection of the first. This yields $x\left(n\right)=2^{n/2}$ if n is even. On the other hand, if n is odd, the central square can be chosen arbitrary, and the last $(n-1)/2$ squares is the reflection of the first $(n-1)/2$ squares. Hence $x(n) = 2\times 2^{(n-1)/2}$ if n is odd. In summary, $x(n) = 2^{\lfloor (n+1)/2\rfloor}$ where $\lfloor\rfloor$ denotes the floor function. The linear recursion (2.1) is readily solved to obtain

$$N(n,m) = 2^{\lfloor (n+1)/2\rfloor}(1-2^{m-1}) + 2^{n+m-1}.$$

For instance this gives 30736 "basic" designs for an 8x8 board. Among these, there are some that are isomorphic up rotations and flips. This leads to the following open question.

Challenge: *How many unique designs **up to rotations and flips** can be made starting with n black and white strips followed by m cross-cuts?*

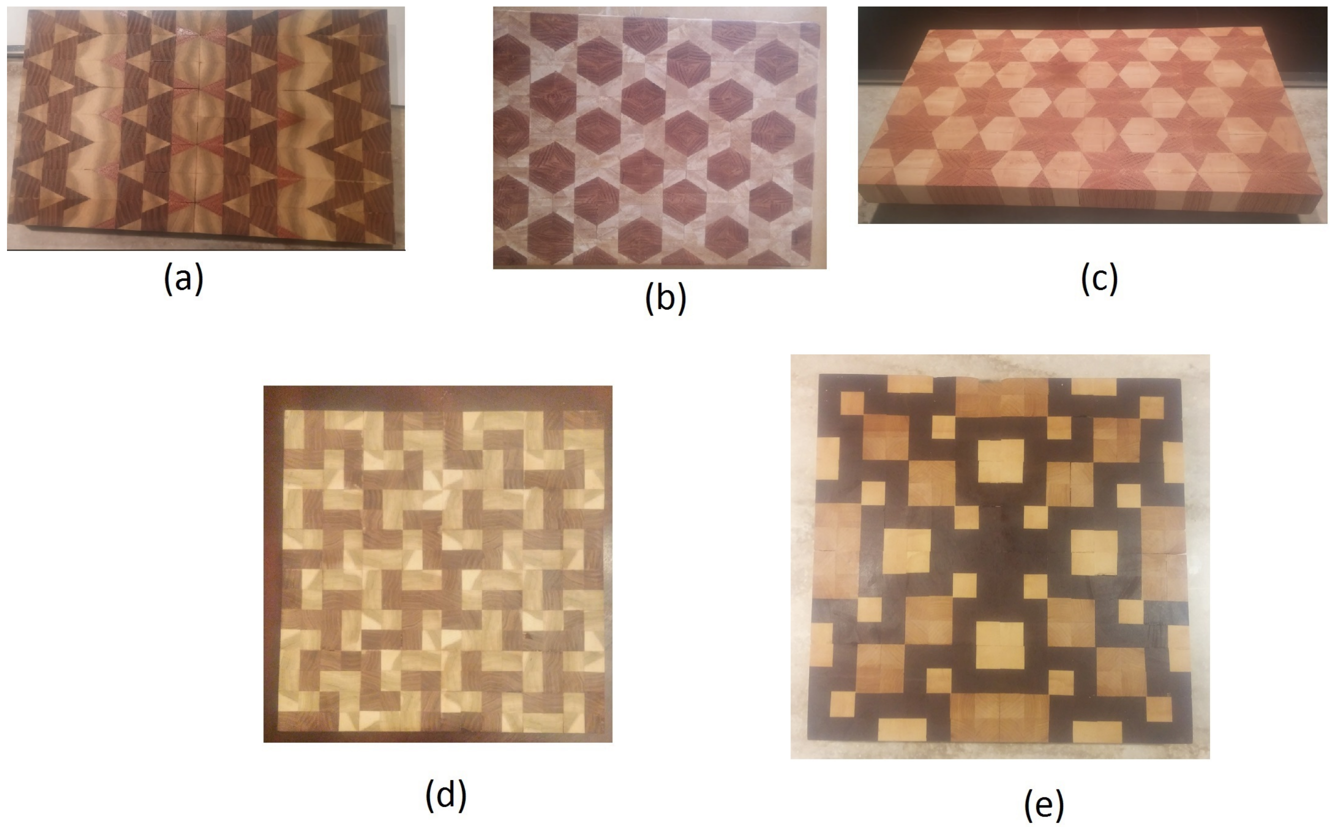


FIG. 3. Designs built by the author, "proving" the techniques discussed in the paper. (a,b,c): using mixed strips, see section 4. (d,e): Fractal designs, see sections 5, 6.

3. CONNECTIONS TO GROUP THEORY

For practical purposes, one rarely uses more than two sets of cut-and-glue operations. Each cut turns some wood into sawdust (the width of a saw blade is about 3mm), so one can quickly eat through alot of wood when making excessively many cuts. Making two sets of cuts ensures that the board is uniform in either vertical in horizontal direction, which increases its stability and prevents bending of the board over time. But further glue-ups do not improve the board's stability.

Nonetheless, for the purposes of making artistic decorative designs as well as writing a math paper we are free to repeat the cutting and glue-up procedure. This naturally leads to the notion of what I will call a "cutting board group", a kind of a Rubic's cube-type game. This group is generated by the following operations on an $n \times m$ array of tiles:

- flip any row or column of tiles;
- swap any two rows;
- swap any two columns.

A natural question is then, what kind of patterns can be obtained using these group operations, starting with black and white strips? If enough cuts can be made, any pattern with the same number of tiles as the initial configuration is producible. More precisely, we show the following:

Theorem 3.1. *Starting with n_1 black and n_2 white rows of an $n \times m$ array of tiles where $n = n_1 + n_2$, it is possible to obtain any array that consists of $n_1 m$ black and $n_2 m$ white tiles in any position using only row/column flips and swaps.*

The proof is constructive. It is a direct consequence of the following lemma.

Lemma 3.2. *Suppose that $m, n \geq 2$. Then any three tiles can be rotated using only flips and swaps.*
Proof of Lemma 3.2.

Step 1: The basic building block is an algorithm (a sequence of flips and swaps) which rotates any three corner tiles of a sub-rectangle. For example, consider the following four operations: flip row 1, flip column 1, flip row 1, flip column 1. This will rotate the top left, top right and bottom left tiles. More generally, given i, j and i', j' with $i' \neq i$ and with $j' \neq j$, the following algorithm rotates tiles (i, j) , (i, j') and (i', j):

$$\begin{array}{ll} \texttt{step 1:} & \texttt{swap rows } i' \texttt{ and } n+1-i. \\ \texttt{step 2:} & \texttt{swap columns } j' \texttt{ and } m+1-j \\ \texttt{step 3:} & \texttt{flip row i} \\ \texttt{step 4:} & \texttt{flip column j} \\ \texttt{step 5:} & \texttt{flip row i} \\ \texttt{step 6:} & \texttt{flip column j} \\ \texttt{step 7:} & \texttt{swap rows } i' \texttt{ and } n+1-i. \\ \texttt{step 8:} & \texttt{swap columns } j' \texttt{ and } m+1-j \end{array} \tag{3.2}$$

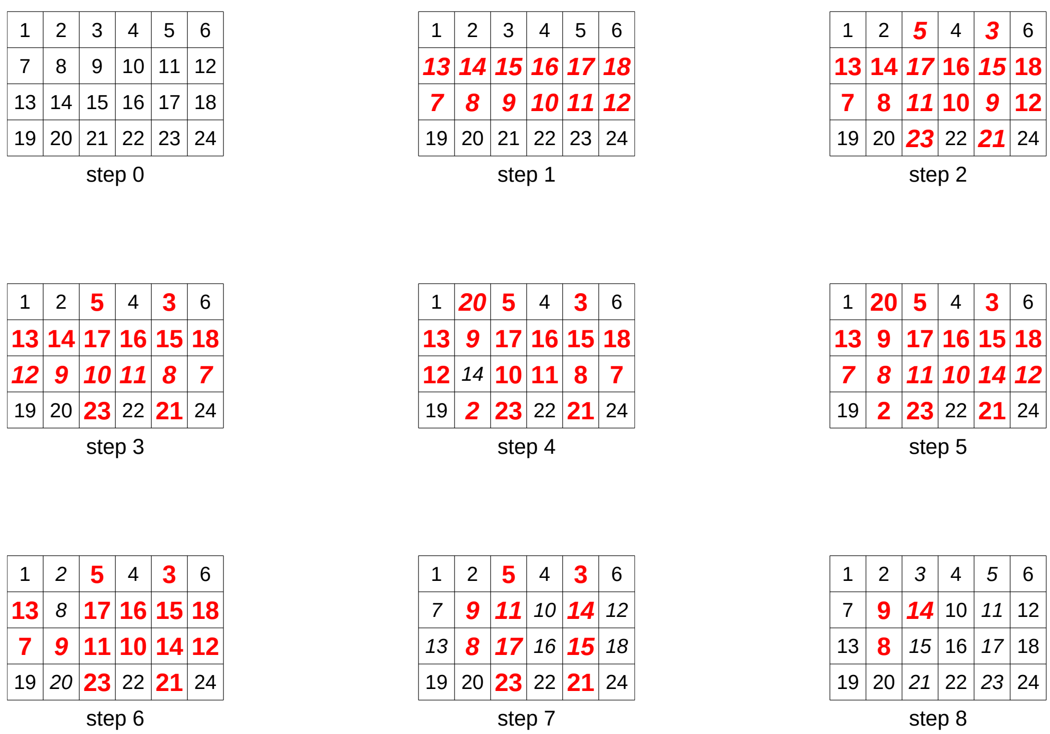
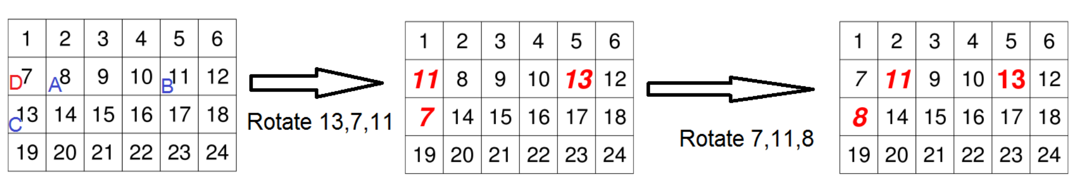


FIG. 4. Executing the sequence of 8 steps given in (3.2) which results in a cyclic permutation of three tiles (i, j), (i, j') and (i', j). Here, $(i, j) = (2, 2), (i, j') = (2, 3), (i', j) = (3, 2)$ which permutes tiles 8,9 and 14. The tiles that change at each step are indicated in italics. Tiles that differ from the original configuration are in red and bold.

This is illustrated in Figure 4.

Step 2: Given two tiles A, B and third tile with arbirary coordinates C we can rotate these as follows. First assume that C is on a different row and column from A and B, and choose a tile D to have the same column as A and same row as B. Apply step 1 to rotate DBC then apply step 1 twice to counter-rotate DAC as illustrated here:

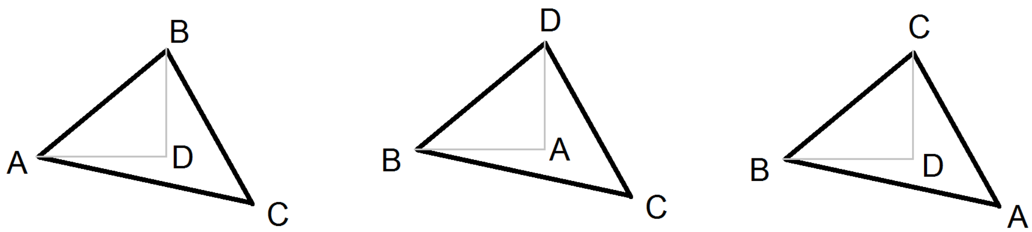


The result rotates the tiles ABC.

If C is on the same row as A and B, the algorithm is the same except choose D to have same column as C but a different row.

Step 3: By transposition, given any three tiles with at least two in the same column, they can be similarly rotated.

Step 4: Finally, given three tiles whose coordinates A, B, C do not share any rows or columns, choose a tile D that is on the same row as A and the same column as B. Apply step 2 to rotate $ADB \to ABD$ then step 3 to rotate $ADC \to DCA$ as shown below:



This rotates arbitrary A, B, C.■

Proof of Theorem 3.1. To complete the proof, we now show that any white and any black tiles can be changed without changing the colour of any other tiles. Given a black tile A and a white tile B, pick a third black tile C and then use the algorithm of Lemma 3.2 to rotate $ABC \to CBA$. This swaps the colours of tiles AB but leaves unchanged the colour of any other tile. Now repeat this procedure to place all black tiles on the top n_1 rows of the array. This produces the initial configuration from any given pattern. Reversing these moves creates the desired pattern from an initial configuration of black and white stripes. ■

A much more difficult question is the following: what is the *minimum* number of moves required to produce a given pattern? Recently, a similar question was answered for Rubic's cube: it was shown in [2] that 20 moves is the "god's number" of moves, that is, any position can be solved using 20 moves or less, and moreover there are positions that require 20 moves. Although upper and lower bounds existed for a long time, it took about 30 years to find the optimal solution.

Define the *"god number"* g to be the minimum number of flips and swaps required to obtain any pattern, starting with n_1 rows of black and $n_2 = n - n_1$ rows of white tiles. **Open question:** determine god's number.

While determining god's number is very difficult, we can readily derive lower and upper bounds. The proof of lemma 3.2 is actually a constructive algorithm and yields an upper bound for the god's number: there are nm tiles in total of which $n_1 m$ are black and the rest are white. It takes at most C moves to switch any white and a black tile (where C is a constant and can be extracted from the lemma; a crude bound is $C = 32 = 4 \times 8$ although this can be improved). To reach a target pattern requires swapping at most all of the black (or white) tiles; which gives an upper bound of $g \leq C \min(n_1, n_2) m$.

To obtain a lower bound, note that at any given configuration, there are a total of $n+m$ flips, and $\binom{n}{2} + \binom{m}{2}$ swaps possible, for a total of $(n^2 + m^2 + n + m)/2$ possible moves. So there are at most $\left((n^2 + m^2 + n + m)/2\right)^k$ configurations after k moves. On the other hand, there is a total of $\binom{nm}{n_1 m}$ possible patterns consisting of $n_1 m$ black tiles. Equating $\left((n^2 + m^2 + n + m)/2\right)^k = \binom{nm}{n_1 m}$ and solving for k yields a lower bound for god's number:

$$g \geq \frac{\log \binom{nm}{n_1 m}}{\log(\frac{n^2+m^2+n+m}{2})}. \tag{3.3}$$

For example if $n = m = 8$ and $n_1 = 1$ we obtain $g \geq 6$. For a "chess-board" $n = m = 8$ and $n_1 = 4$, (3.3) yields $g \geq 10$. More generally, suppose that $m = n$ and suppose that $n_1 = n/2$. we have the asymptotics

$$\log \binom{n^2}{n^2/2} \sim n^2 \log 2, \quad n \to \infty$$

which leads to the following proposition,

Proposition 3.3. *Suppose that $m = n$ and $n_1 = n/2$. Then the god's number is bounded by*

$$C_1 \frac{n^2}{\log(n)} \leq g \leq 32n^2$$

where $C_1 \to \frac{\ln 2}{2}$ as $n \to \infty$.

Similarly, if $m = n$ and $n_1 = 1$ then we have the asymptotics

$$\log \binom{n^2}{n} \sim n \log n$$

which leads to the following proposition,

Proposition 3.4. *Suppose that $m = n$ and $n_1 = 1$. Then the god's number is bounded by*

$$C_1 n \leq g \leq 32n$$

where $C_1 \to 1$ as $n \to \infty$.

In other words, g scales linearly with n when $n_1 = 1$. On the other hand it scales somewhere between $O(n^2/\log n)$ and $O(n^2)$ when $n_1 = n/2$. It is an interesting open question to determine the exact scaling of g in this case.

In the above discussion motivated by cutting boards, we only considered black-or-white tiles. More generally, suppose that a tile is labelled from 1 to nm. What permutations of these unique tiles can be attained using only swaps and flips? The following theorem characterizes this group.

Theorem 3.5. *Consider an $n \times m$ array of unique tiles labelled from 1 to nm. Let G be the group generated by column and row swaps and flips. Suppose that both n and m are divisible by four. Then G is isomorphic to the group of all even symmetric permutations on nm elements. Otherwise it is isomoriphic to the symmetric group of all possible permutation on nm.*

Proof. The group of even permutations is generated by rotating any three elements. Therefore by Lemma 3.2, all even permutations can be obtained by flips and swaps. If n is odd, then swapping two columns is an odd permutation. But adding any odd permutation to the group of even permutations generates all permutations, so in this case the cutting board group is isomorphic to S_{nm}. Similarly if $n \equiv 1, 2, 3 \pmod 4$ then flipping a column is an odd permuation, and similarly for $m \equiv 1, 2, 3 \pmod 4$. On the other hand if both $n, m \equiv 0 \pmod 4$ then any swap or flip is an even permutation and therefore the cutting group is equal to all even permutations. ■

In the case where one of n, m is not divisible by 4, Theorem 3.5 says that any two elements can be swapped. However it is not very constructive: unlike rotating three elements which can be done in $O(1)$ moves independent of n, it is unclear if any two elements can be flipped with $O(1)$ moves. This is another open problem worth exploring.

4. HEXAGONAL PATTERNS

What if we want a hexagonal lattice instead of squares? This requires miter cuts (cuts at an angle). The procedure is illustrated in Figure 5. Start by cutting say 5 black and 5 white strips (figure 5(a). Rip (cut along grain) a 60^0 corner from each, then glue to a strip of opposite colour as in figure 5(b). Rubber bands can be used instead of clamps to glue at this angle. Next, glue the ten strips together as in figure 5(c). I will refer to this as the generator of the pattern. Once dry, make bunch of cross cuts, resulting in identical stripes. Turn them end-grain up, and arrange as shown in figure 5(d).

A huge variety of different boards can be made by using a different order and orientation of the mixed-colour stripes. Figure 6 shows a star-lattice design, a 3d-effect design and several others. All of them are produced in the same way and require the same amount of wood to make. The reader is invited to play with various rules using a javascript applet written by the author [3].

5. FRACTAL DESIGNS

Several standard fractals are amenable to woodworking. Figure 7 shows how to build a Sierpinski triangle without having a CNC router.

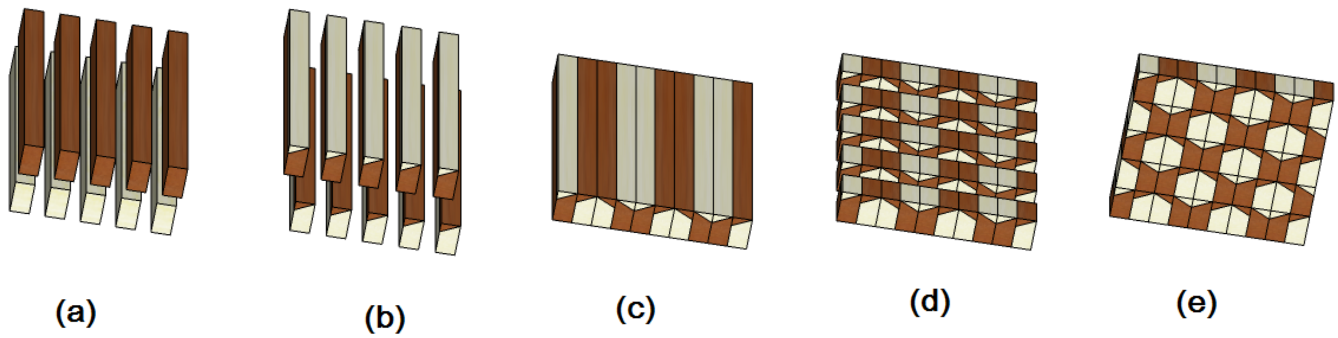


FIG. 5. Making a hexagonal lattice board.

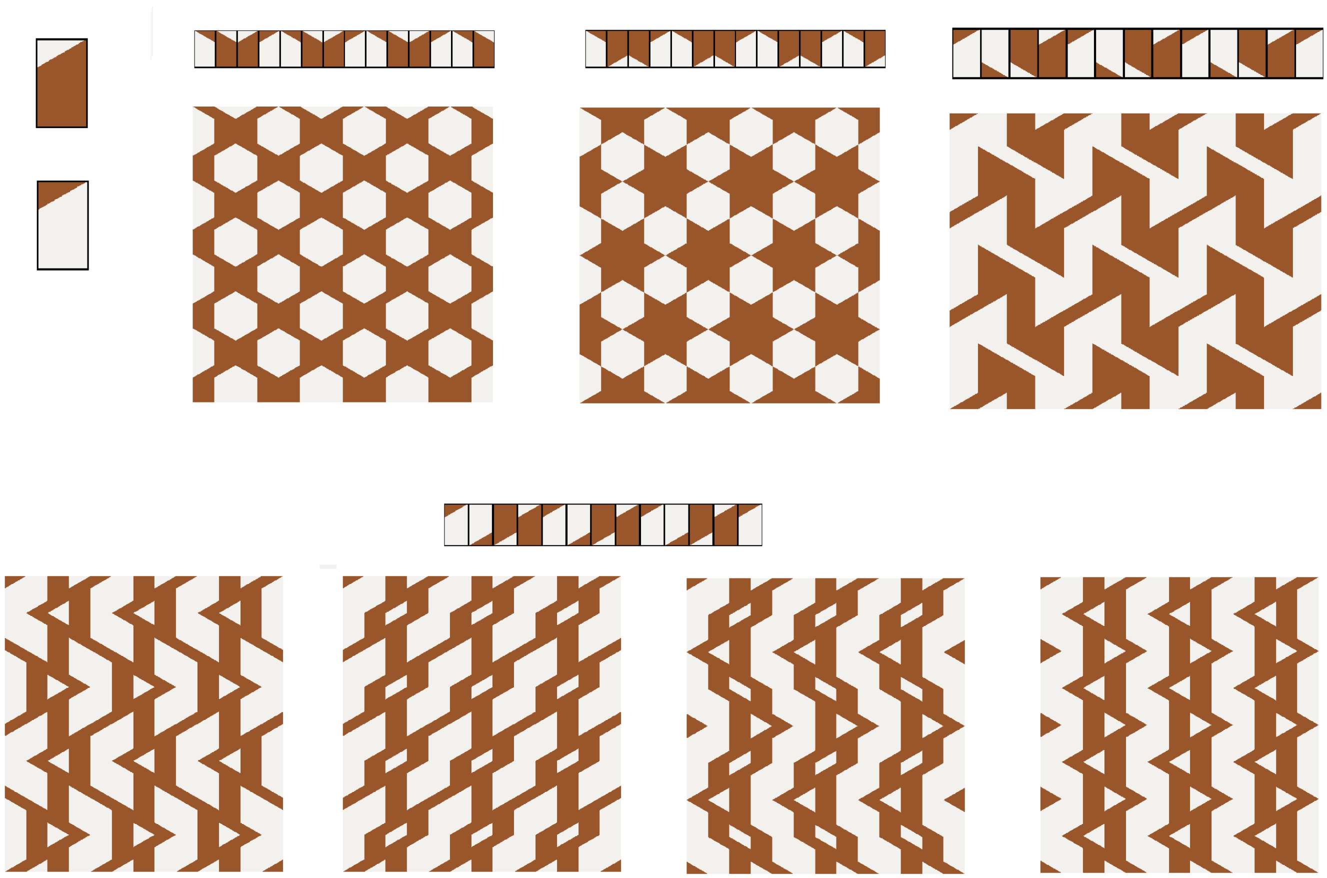


FIG. 6. A small sample of interesting patterns that can be produced with miter cuts using the same method as in Figure 5. The corresponding generators are also shown.

Start with a black square strip. Cross-cut into three equal strips. Also produce a white strip of the same dimensions. Glue them together as shown in figure 7. You obtain a square strip that's 1/3 the length of the original and twice the other dimensions. Rinse and repeat. Each iteration requires two glue-ups (remember that each glue-up should be done along one direction only).

More generally, there is a total of 8 different rigid transformations of each strip possible which preserve its dimensions: rotation by 0, 90, 180 or 270 degrees; or flipping a strip either horizontally, vertically, or along two diagonals.

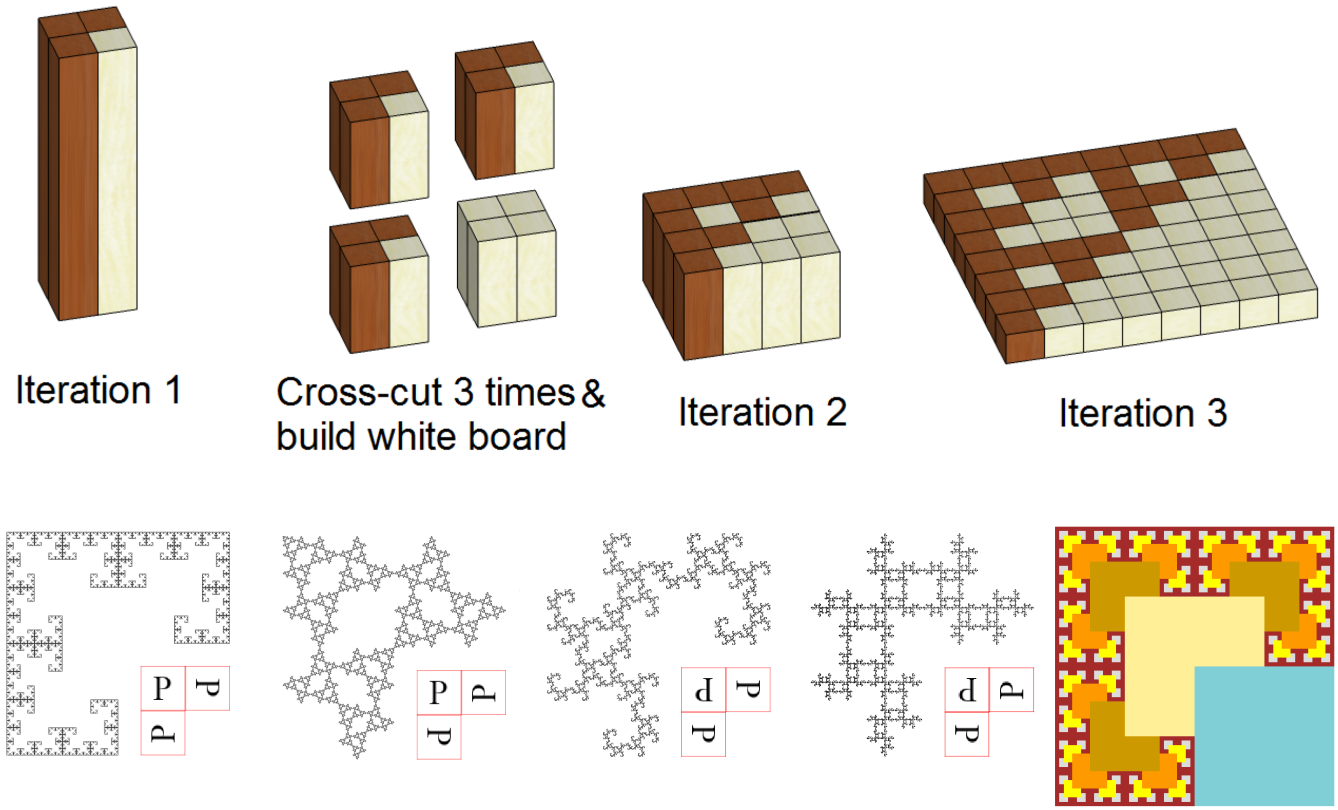


FIG. 7. Top row: building Sierpinsky gasket. Bottom row: other fractals that are buildable, along with the associated rules. The bottom-right has the same rule as the bottom-left, except that different colours are used for a blank board at each iteration.

Labelling these transformations 0 to 7, they are:

$$T_k(z) = \exp\left(i\frac{\pi}{2}k\right) z, \;\; k = 0, 1, 2, 3;$$
$$T_k(z) = T_{k-4}(\bar{z}), \;\; k = 4, 5, 6, 7.$$

Thus, there are a total of $8^3 = 512$ possible fractals that can be produced with this method (assuming the rules for each iteration are the same; of course many more "fractals" are possible if successive iterations use different rules). Figure 7 shows some of them, after 10 iterations. These are easily produced with a few lines of computer code. However even building four iterations poses significant challenge for woodworking, and more than four would be very difficult to do.

One practical complication is that each iteration requires an additional "blank board" to be produced in a separate process. Also, in practice one should start with several black strips separately, so that the first iteration should be done three times in parallel, rather than starting with a single strip. This is because the width of a strip decreases by 3^k after k iterations, so in practice it is difficult to achieve more than 3 iterations (27 cross cuts) out of a single strip of reasonable length.

The limit set of these fractals can be described by iterated function systems (IFS), using three functions that move and transform the three boards that make up each iteration. These functions are:

$$f_1(z) = \frac{1}{2}T_a(z) - \frac{1}{2} + \frac{1}{2}i$$
$$f_2(z) = \frac{1}{2}T_b(z) + \frac{1}{2} + \frac{1}{2}i$$
$$f_3(z) = \frac{1}{2}T_c(z) - \frac{1}{2} - \frac{1}{2}i$$

This rescales the board so that its corners have coordinates $(\pm 1, \pm 1)$. The usual procedure for IFS can be used to produce bottom row of figure 7: start with a random point, choose one of f_1, f_2, f_3 at random, apply it to the point, then repeat a billion times.

To avoid lots of white space, one can vary the colour of the "blank board" inserted at each iteration. An example of this is illustrated in figure 7(bottom right).

Duplicating fractal. A more interesting fractal design can obtained by making four copies at each iteration, instead of making three copies and a blank slate; this also avoids the difficulty of making a new blank slate at each iteration. Start with a square board, call it P. Cut across to make four copies of P. Then re-assemble and glue-up the four boards into a new square board P' that's twice the dimension of the original square. Before assembly, each of the four pieces is transformed in one of eight possible ways (rotations/flips), according to a predetermined rule.

The resulting design depends on the initial board. Figure 8 shows various patterns that are obtained by starting from an initial square half light and half dark: . The reader is invited to try various designs on their own with the help of a javascript applet [3]. There are $8^4 = 4064$ different rules for a given initial board, although many of the rules lead to the same patterns. Overall, it tends to produce more artistic boards – especially at a lower resolution necessary for actual production, say 3 iterations – than the standard iterated function fractal. In part, this is because there is a better mixing of two colours than the iterated map fractal, which is dominated by one colour after a few iterations.

Most of the resulting designs are non-repeating. For example, consider the design shown in 8, top left. Take any row and substitute 0 for light and 1 for dark colour to obtain a sequence of 0's and 1's. For instance the first row of iteration 1 then reads, 0110, the first row of iteration 2 reads 01101001, and so on. The resulting sequence is the well-known and ubiquitous Thue-Morse sequence [4]. This sequence is obtained iteratively, as follows. Start with $s_1 = 0$. Then negate s_1 (i.e. replace any zero by one and any one by zero) and append it to the end of s_1 to obtain $s_2 = 01$; similarly $s_3 = 0110$ and so on. It is an easy exercise to show that the Thue-Morse sequence is aperiodic (in fact transcendental [5, 6] which is much harder to show).

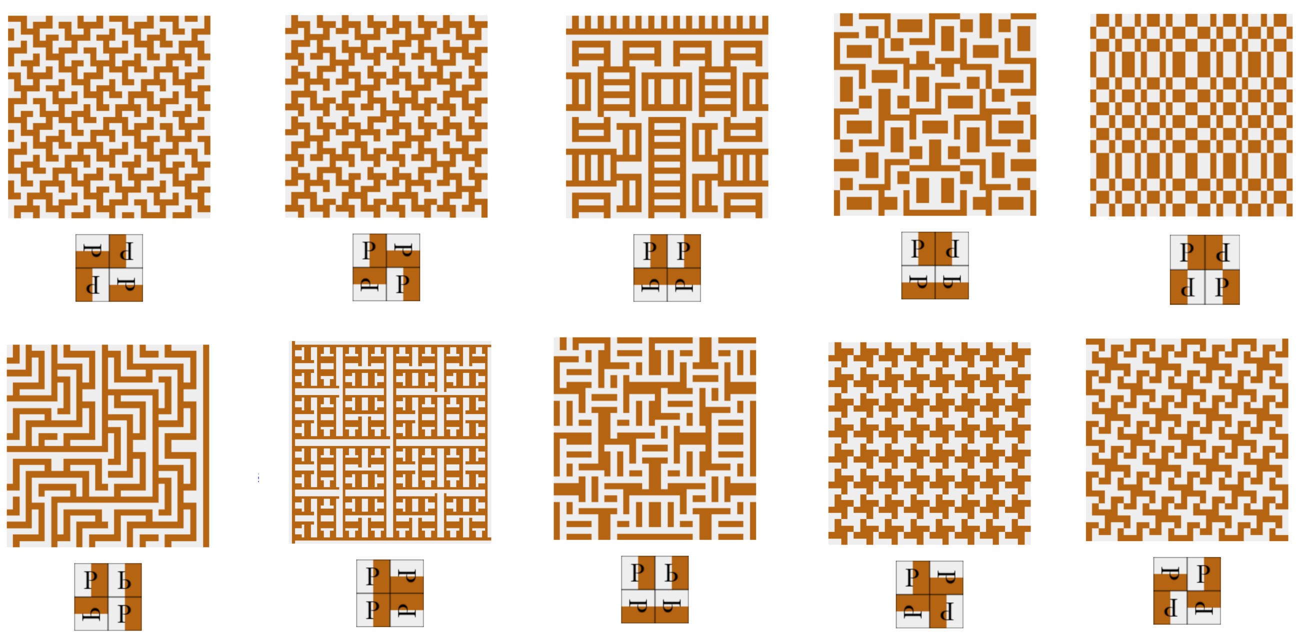


FIG. 8. Duplicating Fractal designs from repeated subdivisions. Here, ten different rules and the resulting pattern (after four iterations) are shown.

Almost all duplicating fractals are non-repeating. Computer experimentation suggests that there are only four distinct periodic patterns that appear with this method. These are: stripe pattern, checker-board pattern, and the two "swastika"-type designs shown on botton right of figure 8. All other patterns do not have any discernable periodicity. This leads to the following open question:

Open question: Starting with initial configuration , which duplicating fractals have periodic patterns? What about more general initial configuration? The conjecture is that the only periodic patterns are the four metioned above.

From artistic point of view, some of the more interesting patterns can be designed by combining the duplicating fractal with IFS fractal as an initial board. An example is shown in Figure 9. It uses two iterations of an IFS fractal as a initial board configuration for the duplicating fractal, then two more iterations of the duplicating fractal.

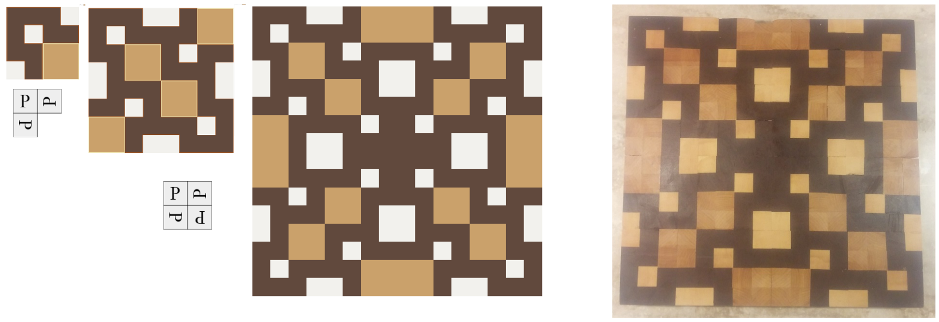


FIG. 9. A combination design. IFS is used for the first two iterations. Then duplicating fractal is used for the last two generations.

6. DOUBLE FRACTAL

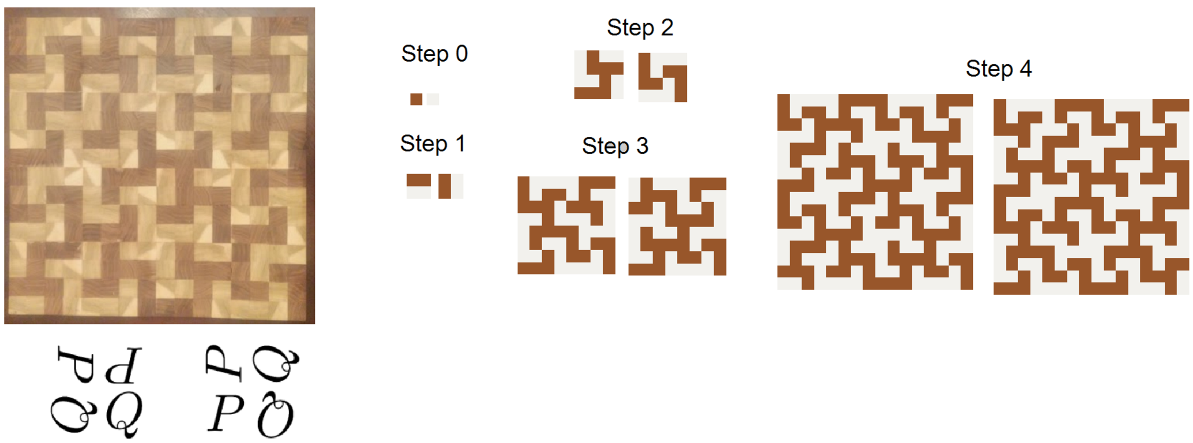


FIG. 10. Creating a double-fractal cutting board. The photograph shows the end-product; four iterations were used. The end-product is supposed to look like the left board of step 4. Looking closely at the top-right corner, a couple of blocks are misplaced due to a mistake in assembly. Mathematical disaster or artistic licence, depending on the point of view.

Finally, we discuss another fractal construction which generalizes both the duplicating fractal as well as the IFS fractals. It creates two boards at the same time. Start with two initial boards say P and Q. They need to be very thick. The board P can be taken all white and the board P can be taken all dark; or more complex initial conditions can be considered. For each iteration, a total of eight copies (some P, some Q) are assembled. For example, cut across to make four copies of P and four copies of Q. Then assemble and glue-up two new boards with twice the dimension from these eight pieces according to a rule $P' = \begin{bmatrix} P & P \\ P & Q \end{bmatrix}$ and $Q' = \begin{bmatrix} P & Q \\ Q & Q \end{bmatrix}$. Rinse and repeat. The resulting pattern is shown in figure 11(top left) after four iterations. As before, we also can rotate/flip the various pieces to create intricate designs. I constructed an example of such a cutting board, see figure 10. See figure 11 for further examples.

Duplicating fractal is a special case of a double fractal, obtained by setting $Q' = P'$. An IFS fractal is also a special case obtained by taking $Q' = \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix}$.

There are eight possible positions for each piece, in addition to 2^8 possible piece arrangements, for a total of $8^8 2^8 \approx 4.3$ billion possible rules, although of course some rules will generate the same (up to rotations/reflections) patterns. The reader is invited to try various designs on their own with the help of a javascript applet [3].

All these fractals are special examples of the Lindenmayer's systems. The board 11(bottom left) appears in photonics literature – see for example [7, 8]. It is a two-dimensional analogue of the Thue-Morse Sequence: its rows and columns form a one-dimensional Thue-Morse sequence.

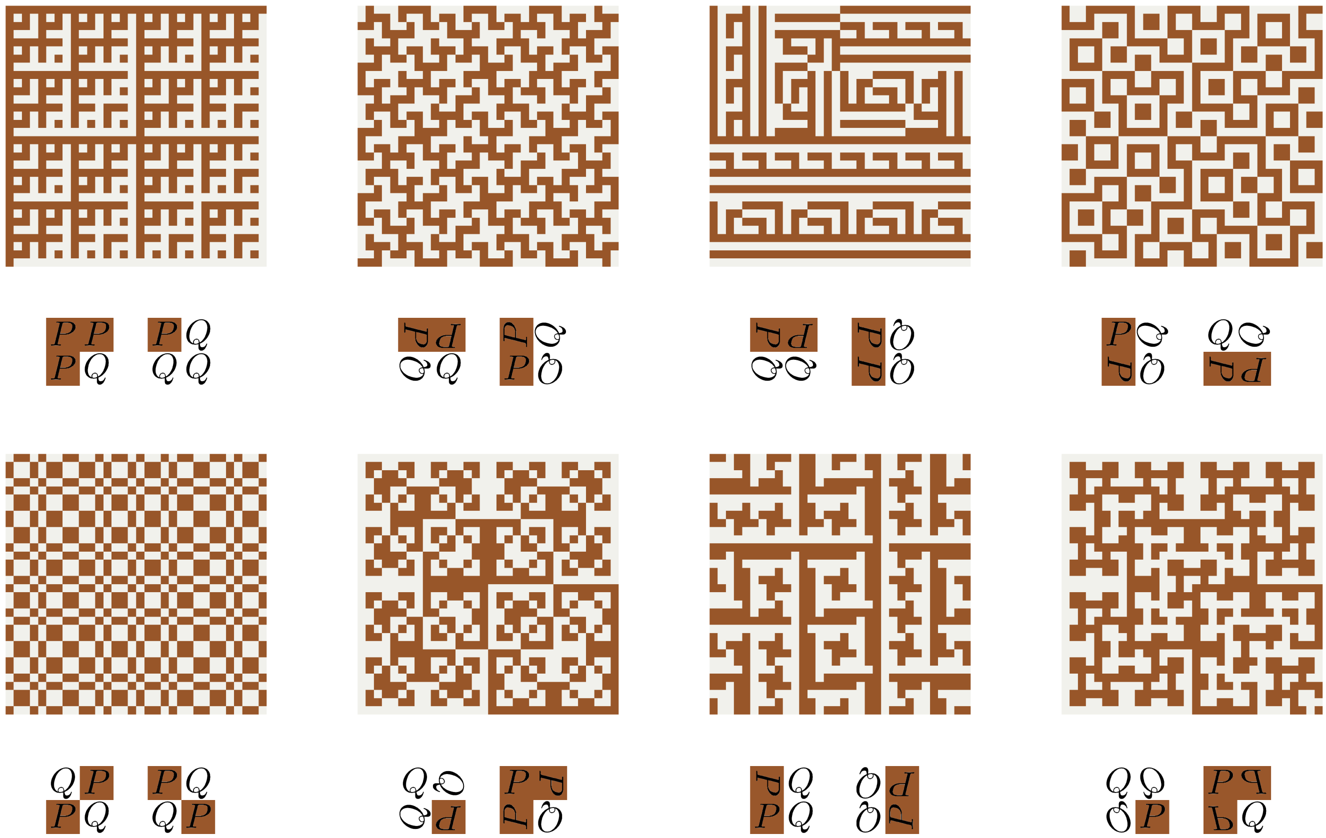


FIG. 11. Examples of various “double-fractal” designs and the associated rules.

7. DISCUSSION

One of my main motivations was to produce artistically beautiful designs in wood. Of course beauty is very subjective. In my opinion, appealing designs often appear at a boundary between order and chaos. This is natural: the design should have some pattern to it in order to spark interest, but not too much lest it becomes boring. This is why fractal designs can be appealing: they have infinite amount of patterns, but none of it is repeating.

FIG. 12. (a) Hakone Yosegi Marquetry veneer. Source: www.hakonemaruyama.co.jp (b) Decorative box with Yosegi veneer marquetry. Source: Wikipedia (c) Detail of an Iranian jewel box with Khatam inlay. Source: Wikipedia. (d) Another example of Khatam. Source: Wikipedia.

Complex mosaics with repeated geometric motifs appear in many cultures throughout the world. Figure 12 shows two such examples made from wood: Hakone Yosegi Marquetry craft from Japan [9], and the art of Khatam from Iran [10]. Hakone marquetry dates back to 18th century. It combines various basic strips of wood into ever-larger blocks, which are eventually planed into paper-thin veneer slices. These slices are in turn used for decorative boxes and furniture. The initial strips typically have a triangular profile and multiple colours are used, leading to very complex patterns. Khatam art uses a similar principle, starting with very thin triangular strips of various colours.

Woodworking provides some natural constraints on the kind of patterns that can be made easily. These constraints raise some interesting and novel questions for mathematicians. I also hope that some of the patterns discussed will spark interest among woodworkers (or inspire mathematicians to try woodworking!).

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