Ring and smoke-ring patterns in Gierer-Meinhardt system

Theodore Kolokolnikov, Xiaofeng Ren and Juncheng Wei

1 Introduction

The Gierer-Meinhardt model is a reaction-diffusion system:

$$\varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0; \qquad \Delta v - v + \frac{u^r}{v^s} = 0$$

with

$$\varepsilon \ll 1.$$

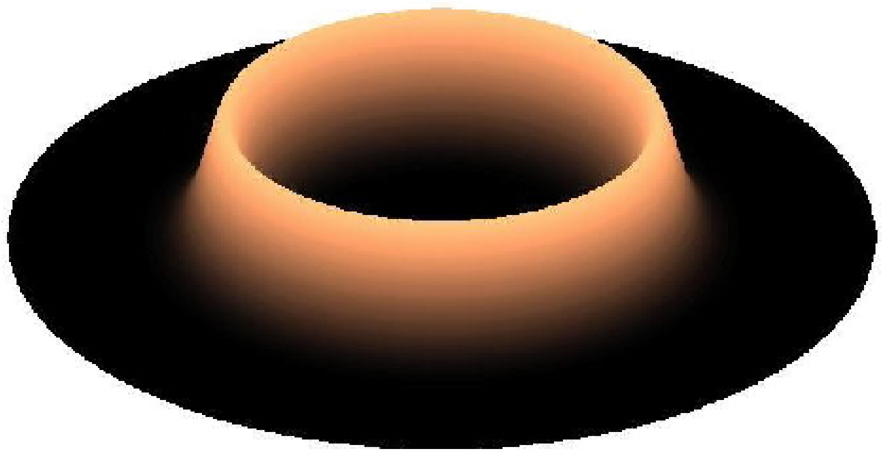
- Comes from mathematical biology (pattern formation in hydra)
- Very popular with mathematicians because it is non autonomous [no max principle, variational formulation] but still can be studied analytically.
- Simplest solutions are spikes; stability analysis very intricate [Doelman, Iron, Kaper, TK, Kowalczyk, Muratov, Ward, Winter, Wei];
- Many other solutions exist: asymmetric spikes [Doelman, Ward, Wei, Winter]
- Generalizations: heterogenous diffusion coefficients [Ward, Wei, Winter]; multiple activators/inhibitors [Wei, Winter]
- What about non-spiky solutions?

2 Ring solutions

Consider *radially symmetric* solutions of GM system inside a ball of radius R:

$$\begin{cases} \varepsilon^2\left(u_{rr}+\frac{N-1}{r}u_r\right)-u+\frac{u^p}{v^q}=0; \\ v_{rr}+\frac{N-1}{r}v_r-v+\frac{1}{\varepsilon}\frac{u^r}{v^s}=0; \\ v'(0)=v'(R)=u'(0)=u'(R)=0 \end{cases}$$

We seek solutions that concentrate on a surface of a sphere of radius r_0. In 2-D they look like this:



Theorem 1: Let

$$M_R(r) := \frac{1}{r}(N-1)\frac{p-1}{q} + \frac{J_1'(r)}{J_1(r)} + \frac{J_{2,R}'(r)}{J_{2,R}(r)}, \tag{1}$$

where

$$J_{2,R}(r) = J_2(r) - \frac{J_2'(R)}{J_1'(R)} J_1(r);$$

and J_1, J_2 satisfy

$$J_{rr} + \frac{N-1}{r} J_r - J = 0$$

with

$$J_2'(0) = 0; \quad J_1(r) \sim \ln(r) \text{ as } r \to 0.$$

Suppose that r_0 satisfies

$$M_R(r_0) = 0.$$

Then there exists a ring-type solution concentrated at the radius $r = r_0$, of the form

$$u(x) \sim Cw\left(\frac{|x| - r_0}{\varepsilon}\right), \quad \varepsilon \to 0$$

where C is some constant and w is the ground state

$$w_{yy} - w + w^p = 0; \quad w \sim Ce^{-|y|}, \; y \to \infty.$$

Remark:

$$J_1(r) = r^{\frac{2-N}{2}} I_\nu(r), \quad J_2(r) = r^{\frac{2-N}{2}} K_\nu(r), \; \nu = \frac{N-2}{2}$$

where I_ν, K_ν are modified Bessel functions of order ν.
Remark: In the case of $N = 3$, J_1, J_2 can be computed explicitly:

$$J_1 = \frac{\sinh r}{r}, \; J_2(r) = \frac{e^{-r}}{4\pi r}. \tag{2}$$

Proof (Standard GM in 2d): In radial variables:

$$\varepsilon^2 u_{rr} + \varepsilon^2 \frac{1}{r} u_r - u + \frac{u^2}{v} = 0; \quad v_{rr} + \frac{1}{r} v_r - \frac{u^2}{\varepsilon} = 0$$

Inner problem:

$$\begin{aligned} r &= r_0 + \varepsilon y; \\ u &= U_0(y) + \varepsilon U_1(y) + \cdots; \quad v = V_0 + V_1(y) + \cdots \end{aligned}$$

Leading order:

$$\begin{aligned} 0 &= U_{0yy} - U_0 + \frac{U_0^2}{V_0}; \quad V_{0yy} = 0 \\ U_0(y) &= \xi w(y); \quad w_{yy} - w + w^2 = 0; \\ V_0(y) &= \xi; \text{ (to be determined later)} \end{aligned}$$

$O(\varepsilon)$ terms:

$$0 = U_{1yy} - U_1 + 2\frac{U_0}{V_0} U_1 - \frac{U_0^2}{V_0^2} V_1 + \frac{1}{r_0} U_{0y}; \quad V_{1yy} + U_0^2 = 0$$

$$V_{1y} = \underbrace{-\int_0^y U_0^2}_{odd} + A \tag{3}$$

$$U_{1yy} - U_1 + 2wU_1 = w^2 V_1 - \xi \frac{1}{r_0} w_y \tag{4}$$

Multiply (4) by w_y and integrate by parts:

$$0 = \int w_y w^2 V_1 - \xi \frac{1}{r_0} \int w_y^2$$

$$\int w_y w^2 V_1 = -A \int \frac{w^3}{3}$$

$$r_0 = -\frac{\xi}{A} \frac{\int w_y^2}{\int \frac{w^3}{3}}.$$

To determine ξ, A we look at the **outer problem**:

$$v_{rr}+\frac{1}{r}v_r-v=-\frac{u^2}{\varepsilon};\qquad \frac{u^2}{\varepsilon}\sim C\delta(r-r_0);\;\; C=\xi^2\left(\int w^2dy\right)=6\xi^2.$$

$$v=6\xi^2G(r,r_0)\;\text{ where }\;\; G_R(r,r_0)=\frac{1}{J_1'(r_0)J_{2,R}'(r_0)-J_1(r_0)J_{2,R}'(r_0)}\begin{cases}J_{2,R}(r_0)J_1(r), & \text{if } r<r_0,\\ J_1(r_0)J_{2,R}(r), & \text{if } r>r_0.\end{cases}$$

Matching: In inner variables:

$$\begin{aligned} r &= r_0+\varepsilon y;\\ v &= 6\xi^2G(r_0^+,r_0)+\varepsilon y6\xi^2\begin{cases}G_r(r_0^+,r_0), & \text{if } y>0\\ G_r(r_0^+,r_0), & \text{if } y<0\end{cases}\end{aligned}$$

$$\begin{aligned} \xi &= 6\xi^2G(r_0,r_0)\\ -\xi^2 3+A &= 6\xi^2G_r(r_0^+,r_0);\qquad \xi^2 3+A=6\xi^2G_r(r_0^-,r_0)\\ A &= 6\xi^2\left(G_r(r_0^+,r_0)+G_r(r_0^-,r_0)\right)\end{aligned}$$

Finally,

$$\frac{1}{r_0}+\frac{J_1'(r_0)}{J_1(r_0)}+\frac{J_{2,R}'(r_0)}{J_{2,R}(r_0)}=0.$$

Theorem 2: Let

$$a = (N-1)\frac{p-1}{q}.$$

Suppose that $N \geq 3$. There are three cases.

(1.a) If $N-2 < a < N-1$ then there exists R_0 such that if $R > R_0$ then $M_R(r) = 0$ has exactly two solutions $0 < r_1 < r_2 < R$, and if $R < R_0$, then $M_R(r) = 0$ has no solution. Moreover, for $R > R_0$, $M_R'(r_1) < 0, M_R'(r_2) > 0$.

(1.b) If $a \geq N-1$ then $M_R(r) = 0$ has no solution for any R.

(1.c) If $a \leq N-2$ then $M_R\,(r) = 0$ has precisely one solution r_1 for any R and moreover $M_R'\,(r_1) > 0$.

Suppose that $N = 2$. Then there exists a number $a_\infty > 1$ whose numerical value is $a_\infty = 1.06119$ such that one of the following holds:

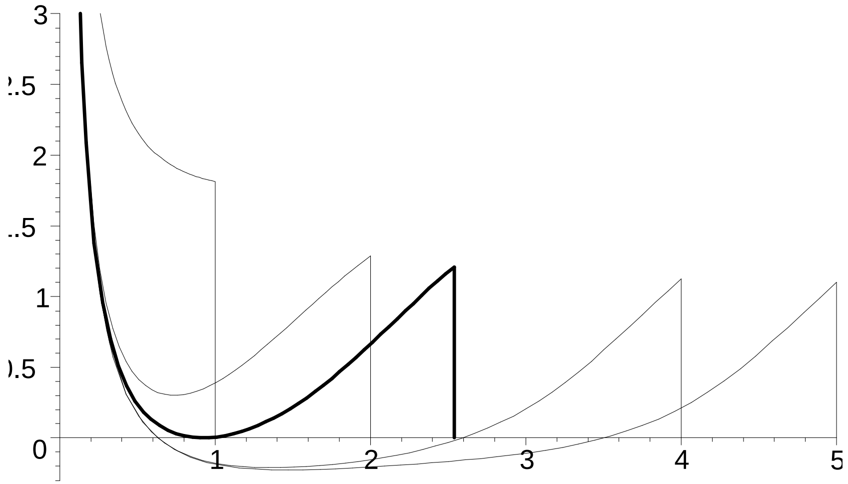
(2.a) If $a \in (0, a_\infty)$ then the situation is the same as in case (1.a).

(2.b) If $a > a_\infty$ then $M_R(r) > 0$ for any R.

(2.c) If $a = a_\infty$ then $M_R(r) > 0$ any $R < \infty$. When $R = \infty$, there exists a number r_0 such that $M_\infty\,(r_0) = 0 = M_\infty'\,(r_0)$, and $M_R(r) > 0$ for any $r \neq r_0$.

As the statement indicates, the situation for $N = 2$ is very different from $N \geq 3$. The case $N = 2$ and $a \in (1, a_\infty)$ has no analogue in higher dimensions and is considerably more difficult.

Sketch of proof ($N \geq 3$**)** Here is M_R for several R values:



Step 1. M_R *is positive for small* R. For small R, expand

$$rM_R(r) \sim a - \left(\frac{1 - r_0^N}{\frac{1}{N-2} + N\frac{r_0^{N-2}}{R^2}} \right), \quad r_0 = \frac{r}{R} \in (0,1)\,; \quad R \ll 1 \tag{5}$$

rhs is $a - N + 2$ when $r = 0$ and increases from there (hence never crosses 0)
Step 2. Since $J'_{2,R}(R) = 0$, it follows that $M_R(R) = \frac{a}{R} + \frac{J'_1(R)}{J_1(R)}$. But J_1 is a strictly increasing and positive function so that $M_R\,(R)$ is always strictly positive.

Step 3. $M_R(r)$ has a double root iff

$$r(J_1(r)J_{2,R}(r))' = -aJ_1(r)J_{2,R}(r); \quad \frac{J_{2,R}'(r)}{J_{2,R}(r)} = -\frac{a}{r} - \frac{J_1'(r)}{J_1(r)} \tag{6}$$

Eliminate R to get:

$$g(r) := (a^2 - a(N-2) - 2r^2)J_1^2(r) + 2raJ_1(r)J_1'(r) + 2r^2 J_1'^2(r) = 0. \tag{7}$$

$g(r)$ satisfies

$$rg' + r^2 C g = J_1^2(r)(B - Ar^2) \tag{8}$$

with

$$A = 4\,(N-1-a)\,, \quad B = (2N-a-4)(a+2-N)a, \quad C = 2N-4-a \tag{9}$$

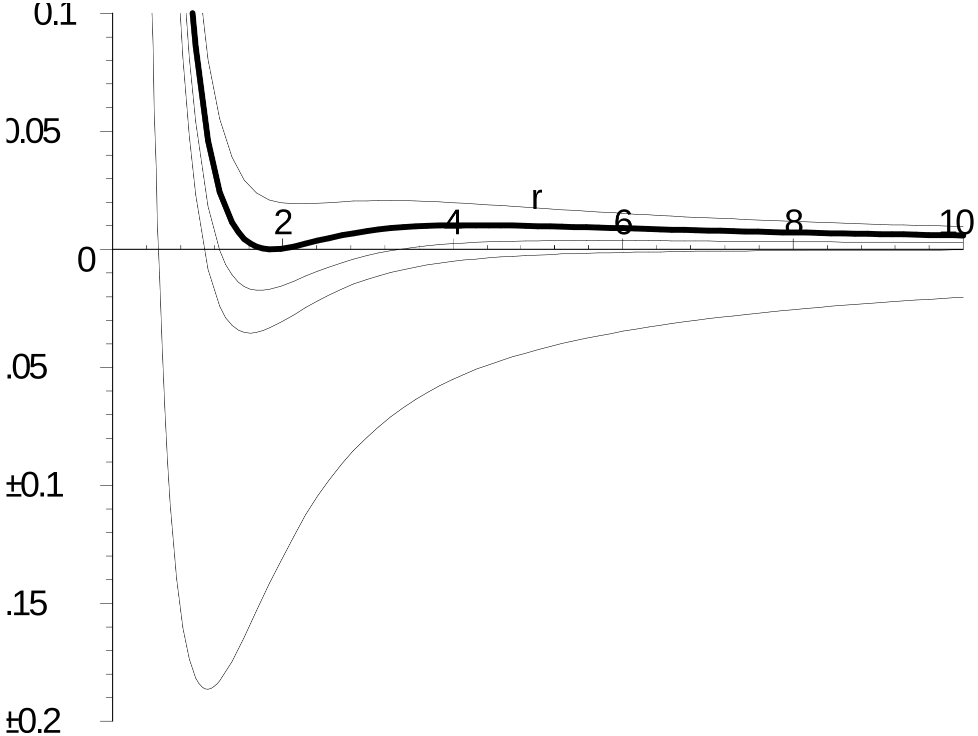
Moreover

$$g(0) = a(a-N+2) > 0; \quad g(\infty) \to -\infty$$

so g has at least one root. Let r_1 be the first root of g. then $g'(r_1) < 0$ so rhs(8)<0. But rhs changes sign only once (and is negative for $r > r_1$); so g cannot have any more roots.
Step 4. For sufficiently large R, M_R has a single root (due to large-argument expansion of $M_\infty(r)$).

The situation is *more complicated* for $N = 2$. Difficult theorem:

- $M_\infty(r)$ has *exactly* 1 root if $0 < a < 1$
- $M_\infty(r)$ has *exactly* 2 roots if $1 < a < a_c = 1.06$
- $M_\infty(r)$ has *no* roots if $a > a_c$.

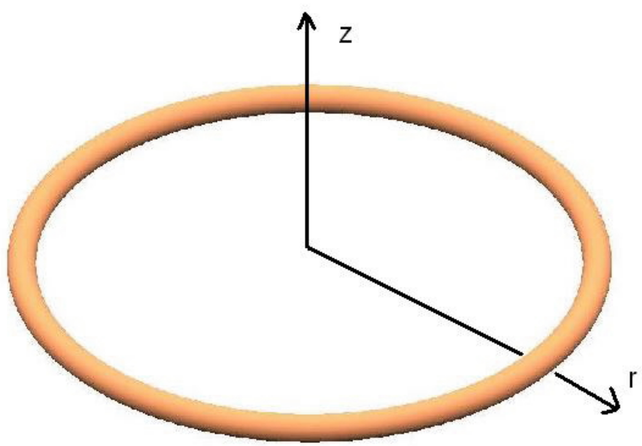


$$a = 0.8, 1, 1.03, a_c, 1.1$$

When $1 \ll R \ll \infty$, there is a sharp transition of r_0 as a crosses 1.

3 Smoke-ring solutions

Consider GM in all of $\mathbb{R}^3$; we seek solutions that concentrate on a *ring.* By taking a cross-section in cylindrical coordinates, this becomes a 2-D problem in (r, z) space:



Define the logarithmic scale:

$$\eta = \frac{-1}{\ln \varepsilon}.$$

Note that we have the relationship

$$0 \ll \varepsilon \ll \eta \ll 1. \tag{10}$$

After proper scaling, the standard GM system is:

$$0 = \varepsilon^2 \left(\Delta_{(r,z)} u + \frac{1}{r} u_r \right) - u + \frac{u^2}{v}; \qquad 0 = \left(\Delta_{(r,z)} v + \frac{1}{r} v_r \right) - v + \frac{\eta}{\varepsilon^2} u^2 \tag{11}$$

Outer problem: u is a spike at $x_0 = (r_0, z_0)$ so we estimate

$$\frac{\eta}{\varepsilon^2} u^2 \sim C\delta(x - z_0), \quad \text{where} \quad C = \int \frac{\eta}{\varepsilon^2} u^2 dx$$

So

$$u = CG(x, x_0)$$

where G is the Green's function which satisfies:

$$\Delta G + \frac{1}{r} G_r - G = -\delta(x, x_0).$$

Descent from 3D: G is a convolution of the 3D Green's function along a ring of radius r_0 :

$$G(x, x_0) = \int_{R^3} \frac{e^{-|x-x'|}}{4\pi|x-x'|} R(x')\, dx'$$

where $R(x')$ is the ring of 2d delta functions:

$$G(r, z, r_0, z_0) = \frac{r_0}{4\pi} \int_0^{2\pi} \frac{\exp[-(r^2 + r_0^2 - 2rr_0 \cos\omega + (z - z_0)^2)^{1/2}]}{4\pi(r^2 + r_0^2 - 2rr_0 \cos\omega + (z - z_0)^2)^{1/2}}\, d\omega \tag{12}$$

After change of variables we have:

$$G = \frac{r_0 e^{-\beta}}{\pi(\alpha - \beta)} \int_0^1 \frac{\exp[-(\alpha - \beta)\tau]\, d\tau}{\sqrt{\tau(\delta + \tau)(1 + \delta + \tau)(1 - \tau)}}, \quad \text{where}$$

$$\beta = [(r - r_0)^2 + (z - z_0)^2]^{1/2}; \quad \alpha = [(r + r_0)^2 + (z - z_0)^2]^{1/2}; \ \delta = \frac{2\beta}{\alpha - \beta} \ll 1;$$

After 7 pages of complicated computations we get the following expansion.

$$G(x_0+\varepsilon y,x_0)=\frac{1}{2\pi\eta}\left[1-\eta\ln R+\eta F_0+\frac{\varepsilon\rho}{2r_0}\left(-1+\eta\ln R+\eta F_1(r_0)\right)+O(\varepsilon^2)\right]$$

where

$$y = \frac{x-x_0}{\varepsilon}=(\rho,Z);\quad R=|y|=\sqrt{\rho^2+Z^2};\ \ \eta=\frac{1}{\ln\left(1/\varepsilon\right)}; \tag{13}$$

$$F_0(r_0) = g_1(2r_0)+\ln 4r_0 \ \ \text{where}\ \ g_1(2r_0)=\int\left(\frac{\exp(-2r_0\tau)}{\tau\sqrt{1-\tau^2}}-\frac{1}{\tau}\right)dt \tag{14}$$

$$F_1(r_0) = 2r_0g_1'(2r_0)-g_1(2r_0)-\ln 4r_0+1 \tag{15}$$

The outer solution in the inner variables becomes:

$$v\sim\xi\left[1-\eta\ln R+\eta F_0+\frac{\varepsilon\rho}{2r_0}\left(-1+\eta\ln R+\eta F_1(r_0)\right)\right],\quad |y|\to\infty$$

where ξ is given by

$$2\pi\xi=\int\frac{u^2}{\varepsilon^2}dx.$$

The smoke-ring radius r_0 will be determined by $O\left(\varepsilon\eta\right)$!! This requires an expanding

$$\xi=\xi_{00}+\eta\xi_{01}+O\left(\varepsilon\right).$$

Inner problem, $y = \dfrac{x - x_0}{\varepsilon}$:

Expand in ε while treating η as a constant:

$$\begin{aligned} u(x,t) &= U = U_0(|y|) + \varepsilon U_1(y) + \cdots \\ V(x,t) &= V = V_0(|y|) + \varepsilon V_1(y) + \cdots \\ \xi &= \xi_0 + \varepsilon \xi_1 + \cdots \end{aligned}$$

The leading order equations are

$$\begin{gathered} 0 = \Delta U_0 - U_0 + \tfrac{U_0^2}{V_0} \\ 0 = \Delta V_0 + \eta U_0^2 \\ 2\pi \xi_0 = \int U_0^2 dx. \end{gathered}$$

Next we expand in η :

$$\begin{aligned} U_0 &= U_{01} + \eta U_{11}; \quad V_0 = V_{01} + \eta V_{11}; \\ \xi_0 &= \xi_{01} + \eta \xi_{01}. \end{aligned}$$

We get

$$\xi_{00} = \frac{1}{\int_0^\infty w^2(s) s ds} = 0.20266 \quad \text{where } \Delta w - w + w^2 = 0$$

After 2 pages of computation,

$$\xi_{01} = \xi_{00} \left(\alpha - 2F_0\right); \quad \alpha = 0.3833$$

This gives us *a correction to* $O\left(\varepsilon\eta\right)$ *term:*

$$\begin{aligned} v \ &\sim \ \left(\xi_{00} + \eta\xi_{01}\right)\left(1 - \eta \ln R + \eta F_0 + \frac{\varepsilon\rho}{2r_0}\left(-1 + \eta \ln R + \eta F_1\right)\right) \\ &= \ \xi_{00}\left(1 - \eta \ln R - \eta\left(F_0 + \alpha\right) + \frac{\varepsilon\rho}{2r_0}\left(-1 + \eta \ln R + \eta\left(F_1 + 2F_0 - \alpha\right)\right)\right) \end{aligned}$$

Next we must study the $O\left(\varepsilon + \varepsilon\eta\right)$ terms... After 4 more pages of solvability computations involving adjoint operator... we finally get the equation for r_0 :

$$F_1(r_0) + 2F_0(r_0) - \alpha + \beta = 0, \quad \text{where } \alpha = 0.3833, \ \ \beta = 0.087$$