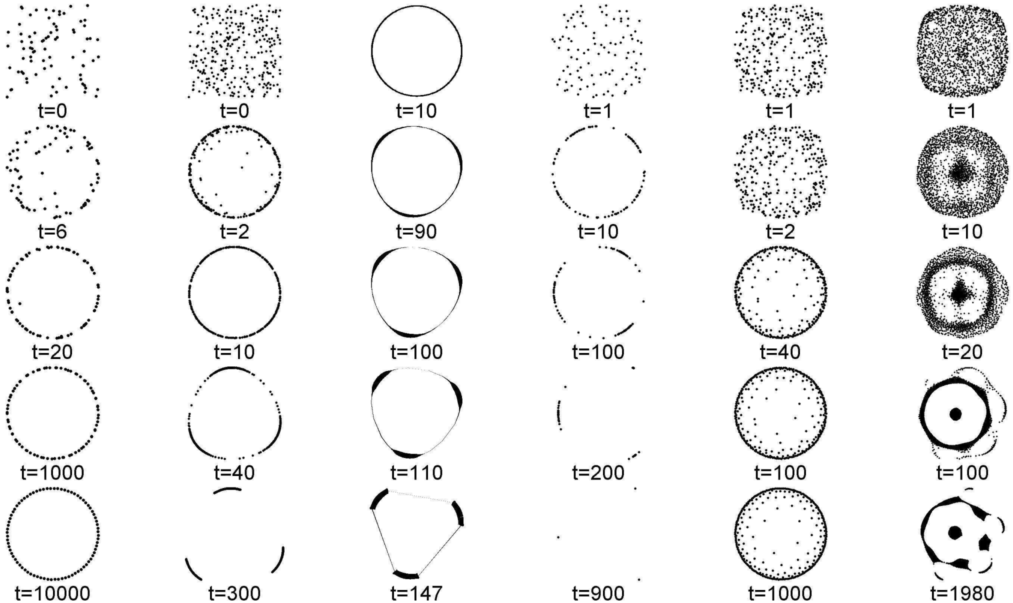
Ring patterns in patrticle aggregation models



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Introduction

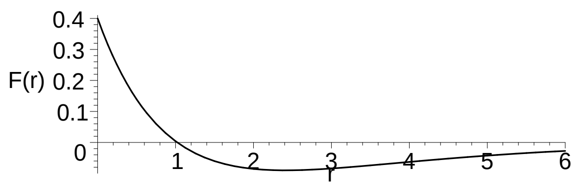
We consider a simple model of particle interaction in 2D

$$\frac{dx_j}{dt} = \frac{1}{N} \sum_{\substack{k=1\ldots N \\ k\neq j}} F\left(|x_k - x_j|\right) \frac{x_k - x_j}{|x_k - x_j|}, \quad j = 1 \ldots N \tag{1}$$

- Models insect aggregation [Edelstein-Keshet et al, 1998] such as locust swarms [Topaz et al, 2008]; robotic motion [Gazi, Passino, 2004].

- Interaction force $F(r)$ is of attractive-repelling type: the insects repel each other if they are too close, but attract each-other at a distance.

- Mathematically $F(r)$ is positive for small r, but negative for large r.



- Commonly, a ***Morse interaction force*** is used:

$$F(r) = \exp(-r) - F\exp(-r/L); \quad F < 1, L > 1 \tag{2}$$

Boundedness, h-stability

- For a **fixed** N, any initial configuration converges to a bounded steady state [GP 2004]

- In the **limit** $N \to \infty$, two possibilities exist: either the particle cloud size grows with N [h-stable case] or its is bounded independent of N [catastrophic regime]. [Ruelle, 1969]

  - H-stable regime: the steady state resembles a hexagonal lattice [Topaz et al, 2006], its diameter is of $O\left(\sqrt{N}\right)$

  - Catastrophic regime: doubling N doubles the density but size and shape is independent of $N \to \infty$.

- Here, we want to take $N \to \infty$, so we are interested in a catastrophic case.

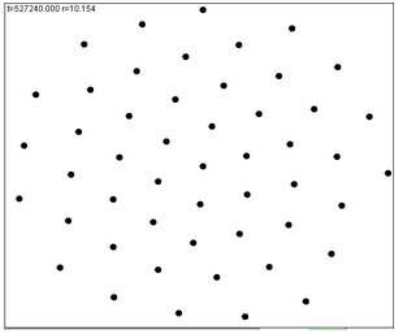
- For Morse interaction force $F(r) = \exp(-r) - F\exp(-r/L)$:

  - In 1D, catastrophic regime if $FL^2 > 1$, else h-stable.

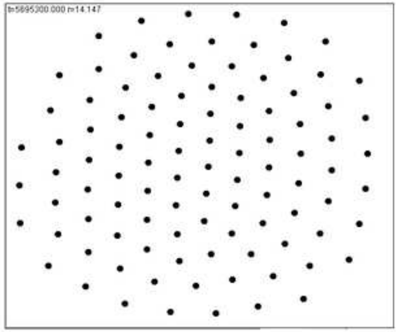
  - In 2D, catastrophic regime if $FL^3 > 1$, else h-stable.

Example of h-stable vs. catastrophic

H-stable, $F(r)=\exp(-r) - 0.7\exp(-0.9r)$

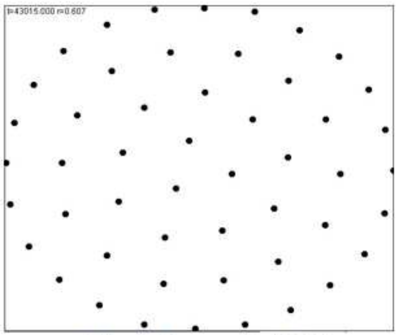


N=50, r=10.15

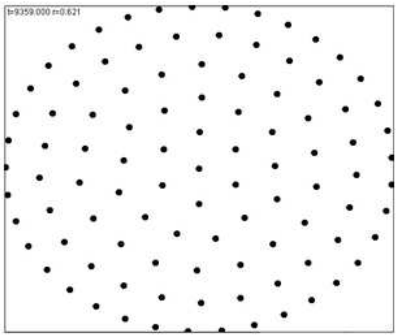


N=100, r=14.15

catastrophic, $F(r)=\exp(-r) - 0.7\exp(-0.5r)$

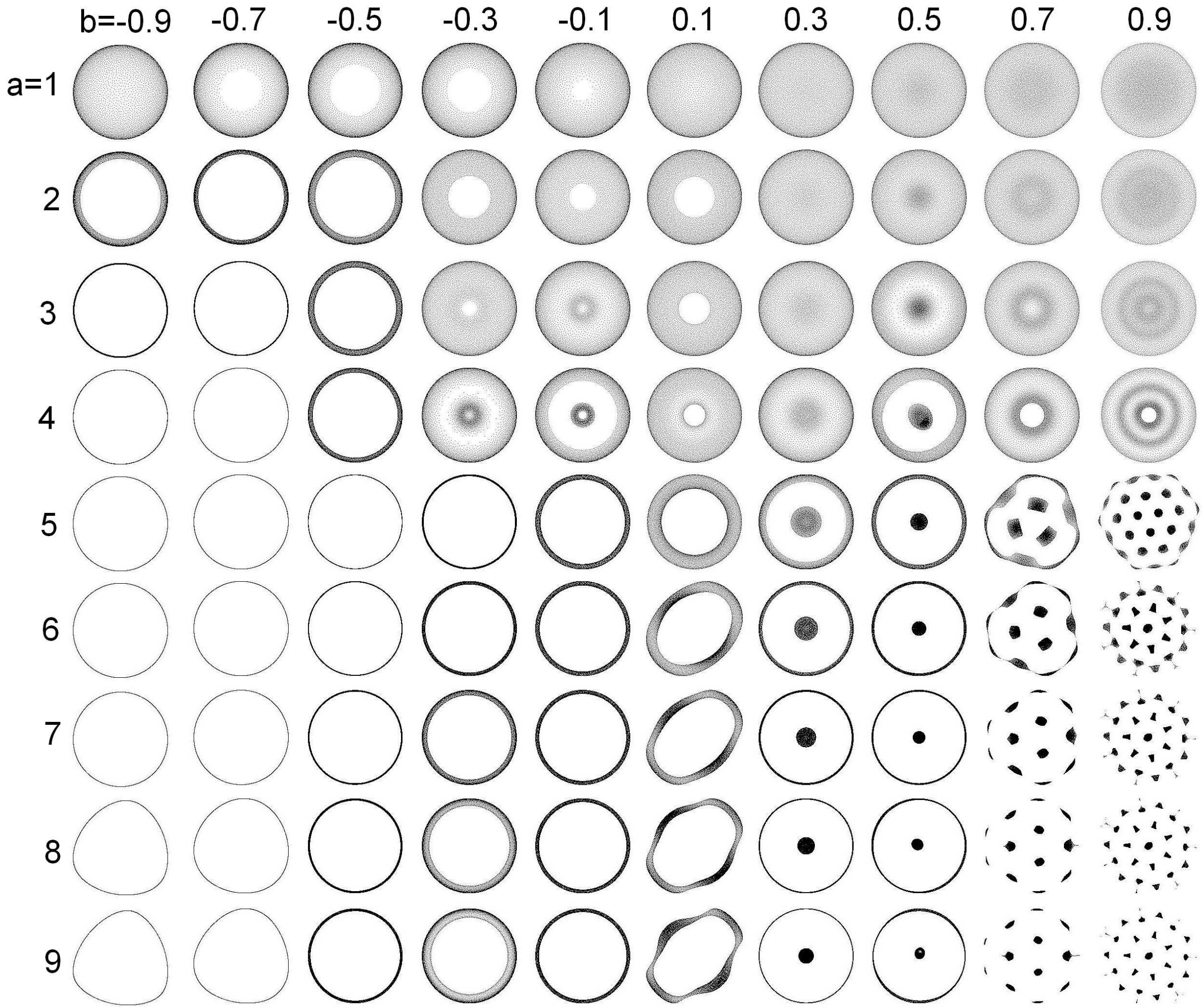


N=50, r=0.607



N=100, r=0.622

Tanh-type force: $F(r) = \tanh\left(\left(1 - r\right)a\right) + b$



Ring-type steady state

- Seek steady state of the form $x_j = r\left(\cos\left(2\pi j/N\right), \sin\left(2\pi j/N\right)\right),\ j = 1 \ldots N$.
- In the limit $N \to \infty$ **the radius of the ring must be the root of**

$$I(r) := \int_0^{\frac{\pi}{2}} F(2r\sin\theta)\sin\theta d\theta = 0. \qquad (3)$$

- For Morse force $F(r) = \exp(-r) - F\exp(-r/L)$, such root exists whenever $FL^2 > 1$ [coincides with 1D catastrophic regime]
- For general repulsive-attractive force $F(r)$, a ring steady state exists if $F(r) \leq C < 0$ for all large r.
- Even if the ring steady-state exists, the time-dependent problem can be ill-posed!

Continuum limit for curve solutions

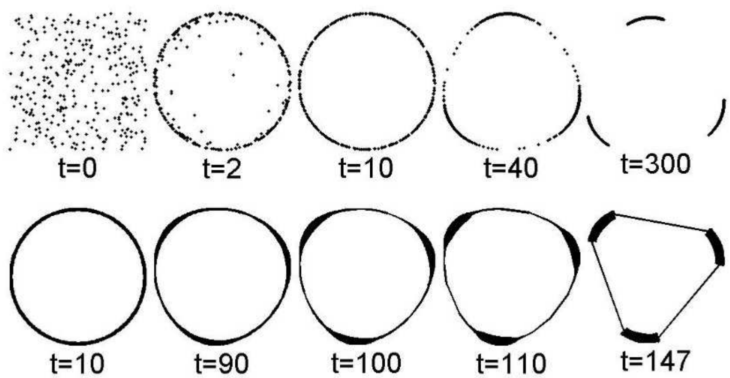
- If particles concentrate on a curve, in the limit $N \to \infty$ we obtain

$$\rho_t = \rho \frac{< z_\alpha, z_{\alpha t} >}{|z_\alpha|^2}; \quad z_t = K * \rho \tag{4}$$

where $z(\alpha; t)$ is a parametrization of the solution curve; $\rho(\alpha; t)$ is its density and

$$K * \rho = \int F\left(|z(\alpha') - z(\alpha)|\right) \frac{z(\alpha') - z(\alpha)}{|z(\alpha') - z(\alpha)|} \rho(\alpha', t) dS(\alpha'). \tag{5}$$

- Depending on $F(r)$ and initial conditions, the curve evolution may be ***ill-defined!***
  - For example a circle can degenerate into an annulus, gaining a dimension.
- We used a Lagrange particle-based numerical method to resolve (4).
  - Agrees with direct simulation of the ODE system (1):



Local stability of a ring

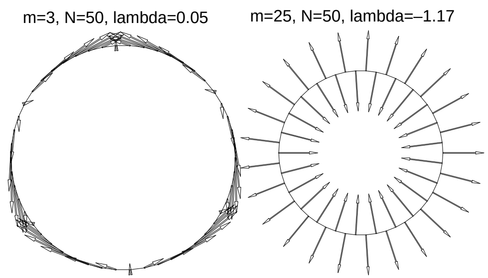
- Linearize: $z(\alpha, t) = r\exp(i\alpha) + \exp(\lambda t)\,\phi(\alpha), \quad \phi \ll 1.$
- Ring is stable of $\mathrm{Re}\,(\lambda) \leq 0$ for all pair (λ, ϕ). There are three zero eigenvalues corresponding to rotation and translation invariance; all other eigenvalues come in pairs due to rotational invariance.
- λ is the eigenvalue of

$$M(m) := \begin{bmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m) \end{bmatrix}; \quad m = 2, 3, \ldots \tag{6}$$

$$I_1(m) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left[\frac{F(2r\sin\theta)}{2r\sin\theta} + F'(2r\sin\theta)\right]\sin^2((m+1)\theta)\,d\theta; \tag{7a}$$

$$I_2(m) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left[\frac{F(2r\sin\theta)}{2r\sin\theta} - F'(2r\sin\theta)\right]\left[\sin^2(m\theta) - \sin^2(\theta)\right]d\theta. \tag{7b}$$

- Eigenfunction is a pure fourier mode when projected to the curvilinear coordinates of the circle.



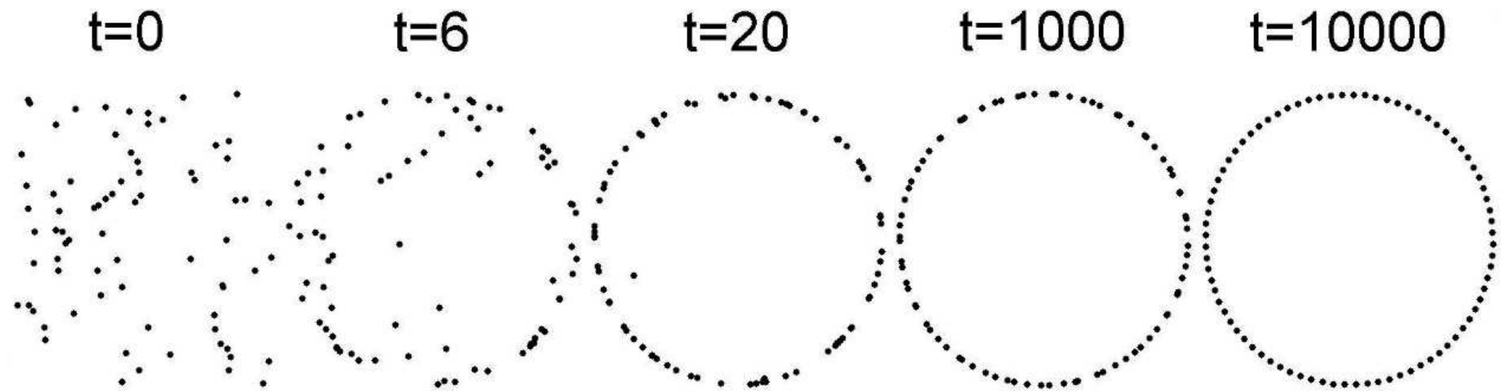
Quadratic force $F(r) = r - r^2$

- Computing explicitly,

$$\text{tr}\, M(m) = -\frac{(4m^4 - m^2 - 9)}{(4m^2-1)(4m^2-9)} < 0, \quad m = 2, 3, \ldots$$

$$\det M(m) = \frac{3m^2(2m^2+1)}{(4m^2-9)(4m^2-1)^2} > 0, \quad m = 2, 3, \ldots$$

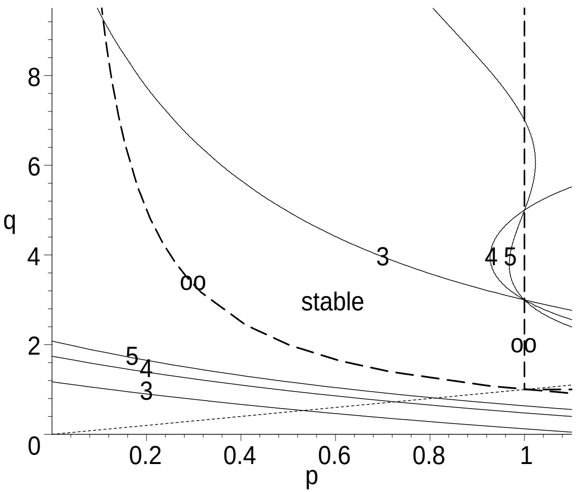
- Conclusion: **ring pattern corresponding to $F(r) = r - r^2$ is locally stable**
- For large m, the two eigenvalues are $\lambda \sim -\frac{1}{4}$ and $\lambda \sim -\frac{3}{8m^2} \to 0$ as $m \to \infty$. The presence of arbitrary small eigenvalues implies the existence of very slow dynamics near the ring equilibrium.

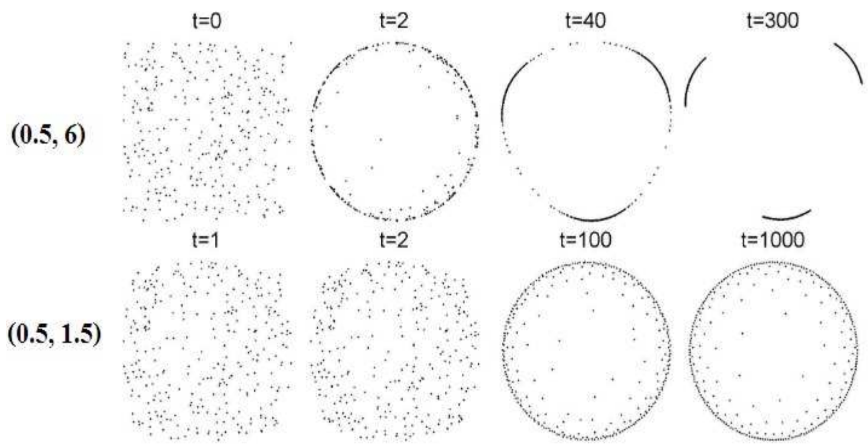


General power force

$$F(r) = r^p - r^q, \;\; 0 < p < q$$

- The mode $m = \infty$ is stable if and only if $pq > 1$ and $p < 1$.
- Stability of other modes can be expressed in terms of Gamma functions.
- The dominant unstable mode corresponds to $m = 3$; the boundary is given by
$$0 = 723 - 594(p+q) - 27(p^2+q^2) - 431pq + 106\left(pq^2+p^2q\right) + 19\left(p^3q+pq^3\right) + 10\left(p^3q^2+p^2q^3\right) + 6\left(p^3+q^3\right) + p^3q^3;$$
- Boundaries for $m = 4, 5, \ldots$ are similarly expressed in terms of higher order polynomials in p, q.





(In)stability of $m \gg 1$ modes

- If $\lambda(m) > 0$ for all sufficiently large m, then we call the ring solution **ill-posed.** Otherwise we call it **well-posed**.

- For ill-posed problems, the ring can degenerate into either an annulus (eg. $F(x) = 0.5 + x - x^2$) or discrete set of points (eg $F(x) = x^{1.3} - x^2$)

- , **if** $F(r)$ **is** C^4 on $[0, 2r]$**, then the necessary and sufficient conditions for well-posedness of a ring are:**

$$F(0) = 0, \quad F''(0) < 0 \text{ and} \tag{8}$$

$$\int_0^{\pi/2} \left(\frac{F(2r\sin\theta)}{2r\sin\theta} - F'(2r\sin\theta) \right) d\theta < 0. \tag{9}$$

- Ring solution for the morse force $F(r) = \exp(-r) - F\exp(-r/L)$ is always ill-posed.

Under construction...

- “Sphere patterns” in 3D and their stability

- What about global stability of rings?

- Forces with sharp transition can produce exotic patterns; examples:
  - Flower: $F(x) =$ max(min(1.6,(1-x)*4),-0.1)
  - Exotic fish: $F(x) =$ max(min(1.6,(1-x)*6),-0.3)
  - Fuzzball: $F(x) =$ max(min(1.6,(1-x)*10),-0.05)