Critical Exponent of a Simple Model of Spot Replication

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Abstract

This paper is concerned with a semilinear elliptic inhomogeneous equation

$$\Delta u - u + (1 + a|x|^q)\, u^p = 0$$

introduced in [C.-C. Chen and T. Kolokolnikov, SIAM J. Math. Anal. 44, no. 5 (2012)] as a simple prototype of self-replication in more complex reaction-diffusion systems. Under certain conditions on p, q, it was previously shown by Chen-Kolokolnikov that the equation has no radial ground state solution when the control parameter a is increased above some threshold. This property is important for the existence of a saddle-node bifurcation proposed in the Nishiura-Ueyema conditions, which is believed to be necessary for an initiation of a self-replication event. In this paper, we generalize Chen-Kolokolnikov's result to non-radial positive solutions by proving a Liouville-type nonexistence theorem. Furthermore we derive a local version of this nonexistence theorem for solutions defined on a bounded ball. Our result indicates that critical values of q derived in [W.-Y. Ding and W.-M. Ni, Arch. Rat. Mech. Analysis, Vol. 91, No.1 (1986)] are also crucial for the existence and nonexistence problem of positive solutions when the space dimension $N \geq 3$.

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Key words. reaction-diffusion equation, spot self-replication, Liouville type theorem, critical exponent

1 Introduction

In this paper we establish a Liouville type theorem for the non-autonomous PDE

$$\Delta u - u + (1 + a|x|^q)\, u^p = 0, \quad u > 0, \quad x \in \mathbb{R}^N. \tag{1.1}$$

This PDE was introduced in [3] as a simple prototype example of spike self-replication that is commonplace in many complex reaction-diffusion systems. These include Gray-Scott model [30, 29, 26, 27, 24, 6, 5, 19, 7], and the related Schnakenberg model [18, 31], the Gierer-Meinhardt model [21, 8, 20], the Belousov-Zhabotinsky reaction [16, 23], the ferrocyanide-iodide-sulfite system [15], the Bonhoffer van-der-Pol-type system [11, 12] and the Brusselator [17].

The inhomogeneity $a\,|x|^q$ is intimately related to self-replication pheonomenon and roughly speaking, models the effect of the "slow" (inhibitor) component of the two-component reaction-diffusion systems with u modelling the "fast" (activator) component which exhibits spike solutions.

In an effort to classify reaction-diffusion systems that can exhibit pulse self-replication, Nishiura and Ueyema in [26] proposed a set of necessary conditions for this phenomenon to occur. Roughly speaking, their conditions can be stated as follows

(S1) The disappearance of the ground-state solution due to a fold point (saddle-node bifurcation) that occurs when a control parameter is increased above a certain threshold value.

(S2) The existence of a dimple eigenfunction at the fold point, which is believed to be responsible for the initiation of the self-replication process. By definition, a dimple eigenfunction is a radially symmetric eigenfunction $\Phi(|x|)$ associated with a zero eigenvalue at the fold point, that decays as $|x| \to \infty$ and that has a positive zero.

(S3) Stability of the steady-state solution on one side of the fold point.

(S4) The alignment of the fold points, so that the disappearance of K ground states, with $K = 1, 2, 3, \ldots$, occurs at roughly the same value of the control parameter.

These conditions are believed to be necessary (although not sufficient) for an initiation of the self-replication event. They were first verified numerically for a certain regime of the Gray-Scott model in [26], [9]. In a different regime, the Gray-Scott model reduces to the so-called core problem [24], [7], [19]. For this core problem, the existence of a fold point (condition (S1)) in one dimension was demonstrated numerically in [24], and conditions (S2), (S3) were also numerically verified in [19]. More recently, the following weaker version of Condition (S1) was shown analytically in [7]:

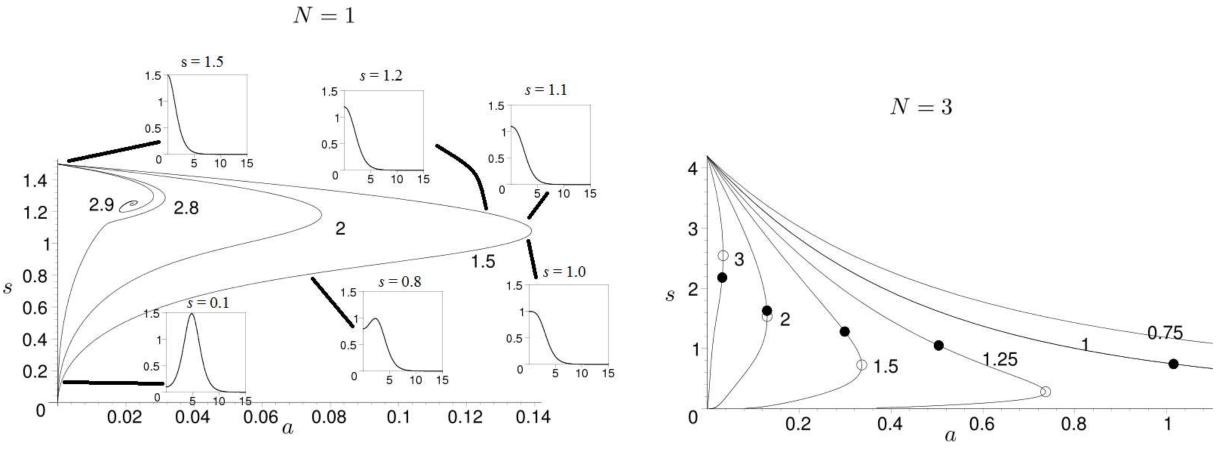
(S1*) The steady-state ceases to exist if a control parameter is increased above a certain threshold value.

In [3] two of the authors of this paper considered a in (1.1) as the control parameter. They showed analytically and numerically that the simple model (1.1) can exhibit self-replication for some values of p and q in any dimension as a is sufficiently increased from zero, due to the dissapearence of the solution at the fold-point. Also conditions (S1*), (S2) and (S3) were analytically verified.

To state the main result in [3] concerning (S1*), we define the critical exponents

$$p^\star = \begin{cases} \dfrac{N+2}{N-2}, & N \geq 3, \\ \infty, & N = 1, 2, \end{cases} \qquad q_c = \frac{(p-1)(N-1)}{2}, \quad q_\star = \frac{(p-1)N}{2}. \tag{1.2}$$

In [3], it was shown that if $p \in (1, p^\star)$ and $q > 0$, then (1.1) always has a positive radial solution when $a > 0$ is small. It is natural to further investigate what happens



(a) (b)

Figure 1: Bifurcation diagram for (1.1) of a vs. $s = u(0)$ with $p = 2$ and for several different values of q as indicated. **(a)** $N = 1$. There is a fold point for all values of q. The bifurcation graph changes its topology at around $q = 2.8$, but is bounded for all q. The inserts show the profile of the steady state $u(r)$ for $q = 1.5, p = 2$ and for s as indicated. **(b)** $N = 3$. Fold point is indicated by an empty circle. Nonradial instability threshold is indicated with filled circle. If $q > 2.1$ then fold-point instability dominates. If $q < 2.1$ then non-radial instability dominates. The fold point exists if $q > 1$; the bifurcation graph is unbounded if $q < 1$. Figure taken from [3].

when a is large. The numerical study in that paper (see Figure 1) indicates that for suitable range of q, the positive radial solution disappears when a is sufficiently large and the bifrucation diagram of the solutions has a fold point. It is precisely this fold point that is responsible for self-replication. Moreover, it was observed that the existence and nonexistence problem when a is large is closely related to an exponent for q proposed by Ding and Ni in [4]. To be more precisely, we quote the main theoretical result in [3] in the following.

Theorem A1 *Given $a \geq 0$, let $u(r)$ with $r = |x|$ be a positive radial solution to (1.1) and let*

$$s := u(0). \tag{1.3}$$

Then the following holds.

(i) *Suppose that $p \in (1, p^\star)$ and $q \geq 0$. Given any constant $a_0 > 0$, there exists a constant $s_0 = s_0(a_0, p, q)$ such that if $0 \leq a < a_0$ then the solution to (1.1) does not exist if $s > s_0$.*

(ii) *Suppose that $p \in (1, p^\star)$ and that either $N \geq 3$ and $q > q_c$ or else $N \leq 2$ and $q > q_\star$. There exists a constant a_1 such the positive solution to (1.1) does not exist if $a > a_1$.*

(iii) *If $N \geq 3$, $0 \leq q < q_c$, then the positive solution to (1.1) exists for all $a \geq 0$, provided that $p \in (1, p^\star)$.*

We note that (ii) implies (S1*). When (i) and (ii) simultaneously hold, the bifurcation graph in the positive (a, s) plane is bounded.

We believe that $q_\star$ for $N = 2$ in (ii) can be replaced by q_c also. However in [3], the authors were unable to prove that. From (ii) and (iii), we see that at least for $N \geq 3$, the exponent q_c is critical for the existence and nonexistence of (1.1) when a is large. To the authors' knowledge, this exponent q_c first appeared in a paper by W.-Y. Ding and W.-M. Ni [4]. One of their results states that if the nonlinear term in a semilinear elliptic equation is radially symmetric and is bounded by $C(1+|x|^q)u^p$, then the equation has a positive radial solution when $N \geq 3, p \in (1, p^\star)$ and $q < q_c$. (iii) is just a direct consequence of this result.

In this paper, we focus on the properties related to Theorem A1. Theorem A1 only deals with the radial case. We are interested in the following question:

Is this Ding-Ni exponent q_c still critical for the non-radial case?

Our answer is affirmative at least for $N \geq 3$. We show that (1.1) has no positive (radial or non-radial) solutions on $\mathbb{R}^N$ when a is large if $N \geq 3$, $p \in (1, p^\star)$ and $q > q_c$ (For $N = 1, 2$, we obtain a weaker result). This nonexistence property can be considered as a Liouville-type theorem depending not only on q but also the magnitude of a. As mentioned, when a is positive and very close to 0, it was shown in [3] that there always exists a radial positive soltion of (1.1) for any $q > 0$ if $p \in (1, p^\star)$. Therefore a non-existence theorem can hold only for a larger a. To prove the Liouville-type theorem for non-radial case, we first obtain an apriori estimate for the positive solutions of (1.1). One difficulty in establishing apriori bounds comes from the fact that (1.1) contains several scaling properties. To over come it, we develop a modified blow-up technique, in which the scaling factor is determined by a balanced relation (see (2.5)). This technique can take care of all the scalings in the equation at the same time and obtain a stronger version of apriori estimate which is indepedent of not only the solution u but also the parameter a. Let $B_s(0) = \{x \in \mathbb{R}^N \mid |x| \leq s\}$. Our result of apriori estimate is as follows.

Theorem 1.1. *Assume $1 < p < p^\star$, $q > q_c$ and $N \geq 1$. Let u be a positive solution of (1.1) on $B_{s+2}(0)$. Then there exists a constant $C_0(p, q)$ depending only on p and q such that*

$$(1 + a|x|^q)u^{p-1}(x) \leq C_0(p, q) \tag{1.4}$$

for $|x| \leq s$.

Now we state our Liouville-type theorem. We give a version on $\mathbb{R}^N$ as well as its local version on a ball with a quantitative estimate of $u(0)$.

Theorem 1.2. *Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $q > q_c$. Then there is $a_0 > 0$ such that for $a \geq a_0$, the following hold.*

(*a*) *(1.1) has no positive solution on* $\mathbb{R}^N$*;*

(*b*) *If* u *is a positive solution of (1.1) on the ball* $B_s(0)$ *with* $s \geq 3$*, then*

$$u(0) \leq C(p,q)s^{-\frac{q-q_c}{p-1}},$$

where $C(p,q)$ *is a constant independent of* u *and* a*. More precisely, we have in this case the estimate*

$$u^2(0) + (q-q_c)\, a\, u^{p+1}(0) \leq [C(p,q)]^2 s^{-\frac{2(q-q_c)}{p-1}}.$$

Remark We note that part (a) in the above theorem follows from part (b) if we let $s \to \infty$ in (b).

For $N = 1, 2$, we have the theorem

Theorem 1.3. *Let* $N = 1$ *or* 2*,* $p > 1$ *and* $q > q_\star$*. Then there is* $a_0 > 0$ *such that for* $a \geq a_0$*, the following hold.*

(*a*) *(1.1) has no positive solution on* $\mathbb{R}^N$*;*

(*b*) *If* u *is a positive solution of (1.1) on the ball* $B_s(0)$ *with* $s \geq 3$*, then*

$$u(0) \leq C(p,q)[(q-q_\star)\, a]^{-\frac{1}{p+1}} s^{-\frac{2(q-q_\star)}{(p+1)(p-1)}},$$

where $C(p,q)$ *is a constant independent of* u *and* a*.*

We conjecture that $q_\star$ can be replaced by q_c in the case $N = 1, 2$. However we do not know how to prove it.

The paper is organized as follows. In section 2, we prove Theorem 1.1, the apriori bound for positive solutions. In section 3, we prove both Theorem 1.2 and Theorem 1.3.

2 Apriori estimate

To prove Theorem 1.1, we need the following two Liouville-type theorems obtaind by Gidas and Spruck [10] and Bianchi [1] respectively.

Theorem A2 (Gidas and Spruck) *Assume* $1 < p < p^\star$*. Then* $u = 0$ *is the only nonnegative solution of*

$$0 = \Delta u + u^p \text{ in } \mathbb{R}^N. \tag{2.1}$$

Theorem A3 (Bianchi) *Assume* $1 < p < p^\star$ *and* $q \geq 0$*. Then* $u = 0$ *is the only nonnegative solution of*

$$0 = \Delta u + (\alpha + \beta|x|^q)u^p \text{ in } \mathbb{R}^N, \tag{2.2}$$

where $\alpha \geq 0, \beta \geq 0$ *and* $\alpha^2 + \beta^2 > 0$*.*

Theorem A3 is a generlization of Theorem A2. Both of them were proved for $N \geq 3$. However, the same conclusion for $N = 1, 2$ follows from the property that there exists no nonnegative super harmonic function on $\mathbb{R}^2$ or $\mathbb{R}^1$ except constant functions.

Proof of Theorem 1.1. We prove the theorem by contradiction. Suppose there exist sequences $\{u_k\}$, $\{a_k\}$ and $\{x_k\}$ such that u_k is a solution of (1.1) with $a = a_k \geq 0$, $|x_k| \leq s$ and

$$(1 + a_k|x_k|^q)u_k^{p-1}(x_k) \to \infty \text{ as } k \to \infty. \tag{2.3}$$

In the following, we employ an blow-up argument to obtain a positive solution of (2.1) or more generally, (2.2), which lead to a contradiction to Theorem A2 or Theorem A3. To do this, we need to choose a point z_k near x_k such that z_k behaves like a local maximum point of u_k in a suitable sense and change the variable x with the natural scaling factor

$$\hat{L}_k = [(1 + a_k|z_k|^q)u_k^{p-1}(z_k)]^{1/2}. \tag{2.4}$$

However equation (1.1) has several different scaling properties. To deal with them in a unified way and take care of the possible unboundedness of a_k, we consider a modification L_k of $\hat{L}_k$, which is determinated by the relation

$$L_k = \{[1 + a_k(|z_k| + \frac{1}{L_k})^q]u_k^{p-1}(z_k)\}^{1/2}. \tag{2.5}$$

To see that how z_k is chosen and such an L_k exists, we let

$$\hat{L}_{k,x} = [(1 + a_k|x|^q)u_k^{p-1}(x)]^{1/2}$$

and consider the function

$$F_{k,x}(L) = \{[1 + a_k(|x| + \frac{1}{L})^q]u_k^{p-1}(x)\}^{1/2}, L > 0. \tag{2.6}$$

Since $F_{k,x}(L)$ is strictly decreasing in L, $\lim_{L\to 0+} F_{k,x}(L) = \infty$, and $\lim_{L\to\infty} F_{k,x}(L) = \hat{L}_{k,x}$, we conclude that there is a unique $L_{k,x} > 0$ such that $L_{k,x} = F(L_{k,x})$. Moreover we have $\hat{L}_{k,x} < L_{k,x} < F(\hat{L}_{k,x})$.

Now we describe how to choose z_k. Let $B_k = \{x : |x - x_k| \leq 1\}$ and let $d(x, \partial B_k)$ denote the distance between x and ∂B_k. We define M_k as follows and choose z_k to be a point which achieves M_k:

$$\begin{aligned} M_k &= \max_{x\in B_k} d(x, \partial B_k)^2 L_{k,x}^2 \\ &= d(z_k, \partial B_k)^2 L_{k,z_k}^2. \end{aligned} \tag{2.7}$$

By (2.3),

$$\begin{aligned} L_{k,z_k} &= M_k^{1/2} d(z_k, \partial B_k)^{-1} \\ &\geq M_k^{1/2} \\ &> [d(x_k, \partial B_k)^2 L_{k,x_k}]^{1/2} \\ &\geq [d(x_k, \partial B_k)^2(1 + a_k|x_k|^q)u^{p-1}(x_k)]^{1/2} \\ &= [(1 + a_k|x_k|^q)u^{p-1}(x_k)]^{1/2} \to \infty \end{aligned} \tag{2.8}$$

as $k \to \infty$.

In the following, we denote

$$L_k = L_{k,z_k}$$

and have (2.5) since $L_{k,z_k} = F(L_{k,z_k})$. Now we take the scaling $y = L_k\,(x - z_k)$ and let

$$v_k(y) = u_k^{-1}(z_k)u_k(x). \tag{2.9}$$

Then v_k satisfies $v_k(0) = 1$ and

$$0 = \Delta_y v_k - L_k^{-2} v_k + A_k(y) v_k^p \ \text{ in } \ \mathbb{R}^N, \tag{2.10}$$

where

$$A_k(y) = \frac{1 + a_k|x|^q}{1 + a_k(|z_k| + \frac{1}{L_k})^q} \text{ with } y = L_k\,(x - z_k).$$

We consider (2.10) in the region $|x - z_k| \le 1/2d(z_k, \partial B_k)$, i.e., $|y| = L_k|x - z_k| \le 1/2M_k^{1/2}$. Note that this region $|x - z_k| \le 1/2d(z_k, \partial B_k)$ is contained in $B_{s+1}(0)$. For a fixed $R > 1$ with $R < 1/2M_k^{1/2}$, we further restrict (2.10) to the domain $|y| < R$, that is, the domain $|x - z_k| < RL_k^{-1}$ in the variable x. In this range, by $|x - z_k| < RL_k^{-1} \le \frac{1}{2}d(z_k, \partial B_k) \le 1/2$ and (2.7), we have

$$d(x, \partial B_k) \ge d(z_k, \partial B_k) - |x - z_k| \ge \frac{1}{2}d(z_k, \partial B_k)$$

and

$$d(z_k, \partial B_k)^2 L_k^2 = M_k \ge d(x, \partial B_k)^2 L_{k,x}^2 \ge \frac{1}{4}d(z_k, \partial B_k)^2 L_{k,x}^2,$$

which implies

$$L_{k,x} \le 2L_k \tag{2.11}$$

Also we have

$$v_k(y) = u_k^{-1}(z_k)u_k(x) \le \{\frac{4[1 + a_k(|z_k| + \frac{1}{L_k})^q]}{1 + a_k(|x| + \frac{1}{L_{k,x}})^q}\}^{\frac{1}{p-1}} =: H(k, y). \tag{2.12}$$

In this range ($|y| < R$ and $|x - z_k| < RL_k^{-1}$), we further show $v_k(y)$ and $A_k(y)$ are uniformly bounded in k. First by the inequality

$$\frac{e + f}{g + h} \le \frac{e}{g} + \frac{f}{h} \tag{2.13}$$

for positive e, f, g, h, we have

$$A_k(y) = \frac{1 + a_k|z_k + \frac{y}{L_k}|^q}{1 + a_k(|z_k| + \frac{1}{L_k})^q} \le \frac{1 + a_k R^q(|z_k| + \frac{1}{L_k})^q}{1 + a_k(|z_k| + \frac{1}{L_k})^q} \le 1 + R^q. \tag{2.14}$$

For the estimate of $v_k(y)$, we divide it into two cases.
Case 1. $|z_k|L_k \le 2R$: By inequality (2.13) and (2.11), we have

$$\begin{aligned} H(k,y) &\le \{\frac{4[1+a_k(|z_k|+\frac{1}{L_k})^q]}{1+a_k(\frac{1}{2L_k})^q}\}^{\frac{1}{p-1}} \\ &\le \{\frac{4[1+a_k(\frac{2R+1}{L_k})^q]}{1+a_k(\frac{1}{2L_k})^q}\}^{\frac{1}{p-1}} \le \{4(1+(4R+2)^q)\}^{\frac{1}{p-1}}. \end{aligned} \tag{2.15}$$

Case 2. $|z_k|L_k \ge 2R$:
From $|x| \ge |z_k| - |x - z_k| \ge |z_k| - RL_k^{-1}$, it follows

$$\begin{aligned} H(k,y) &\le \{\frac{4[1+a_k(|z_k|+\frac{1}{L_k})^q]}{1+a_k(|z_k|-\frac{R}{L_k}+\frac{1}{2L_k})^q}\}^{\frac{1}{p-1}} \\ &\le \{\frac{4[1+a_k(|z_k|+\frac{1}{L_k})^q]}{1+a_k(\frac{1}{2}|z_k|+\frac{1}{2L_k})^q}\}^{\frac{1}{p-1}} \le [4(1+2^q)]^{\frac{1}{p-1}}. \end{aligned} \tag{2.16}$$

Inequalities (2.15) and (2.16) together imply that $v_k(y) \le C(R)$ for some $C(R)$ independent of k if $|y| < R$.

The argument above shows that A_k and v_k are bounded on any compact set. Therefore we can obtain a subsequence of v_k, still denoted by v_k, via a diagonal process such that v_k converges in $C^{2,\alpha}_{loc}$ to some v on $\mathbb{R}^N$. Moreover v satisfies

$$0 = \Delta v + A(y)v^p \ \text{ in } \ \mathbb{R}^N, \ v \ge 0, \ v(0) = 1 \tag{2.17}$$

for some function $A(y)$. After further passing to a subsequence of v_k if necessary, $A(y)$ takes the form

$$A(y) = \tau + |y_0 + \sigma y|^q, \tag{2.18}$$

where

$$\begin{aligned} \tau &= \lim_{k\to\infty} \frac{1}{1+a_k(|z_k|+\frac{1}{L_k})^q}, \\ \sigma &= \lim_{k\to\infty} \frac{a_k^{1/q}L_k^{-1}}{[1+a_k(|z_k|+\frac{1}{L_k})^q]^{1/q}}, \\ y_0 &= \lim_{k\to\infty} \frac{a_k^{1/q}z_k}{[1+a_k(|z_k|+\frac{1}{L_k})^q]^{1/q}} \in \mathbb{R}^N, \end{aligned}$$

When $\sigma = 0$, $A(y)$ is a positive constant. When $\sigma \ne 0$, $A(y)$ has the form $\tau + |\sigma\tilde{y}|^q$ with $\tilde{y} = y + \sigma^{-1}y_0$. Therefore (2.17) together with (2.18) contradicts either Theorem A2 or Theorem A3. The proof is complete. □

Remark. In (2.18), we have $0 \le \tau \le 1, 0 \le \sigma \le 1$ and $|y_0| \le 1$. The relative asymptotic magnitudes of 1, $a_k|z_k|^q$ and $a_kL_k^{-q}$ determine τ, σ and y_0. The following are some examples showing this: $A(y) = 1$ if $\lim_k |z_k|L_k = \infty$ and $\lim_k a_k|z_k|^q = 0$; $A(y) = |y|^q$ if $\lim_k |z_k|L_k = 0$ and $\lim_k a_kL_k^{-q} = \infty$; and $A(y) = \tau + |y_0 + \sigma y|^q$ for some $\tau > 0$, $\sigma > 0$ and $y_0 \ne 0$ if $0 < \lim_k |z_k|L_k < \infty$ and $0 < \lim_k a_k|z_k|^q < \infty$.

3 Liouville type theorems

In this section, we first derive a generalized Pohozave identity for a solution of (1.1) on $B_R := \{x \in \mathbb{R}^N \mid |x| \le R\}$.

Lemma 3.1. *If u is a solution of (1.1), then*

$$\begin{aligned}
&\int_{B_R} \frac{k(N+k)}{2} u(x\cdot\nabla u) r^{k-2}\,dx \qquad (3.1)\\
= &\int_{\partial B_R} r^k \left[(x\cdot\nabla u + \frac{N+k}{2}u)\frac{\partial u}{\partial \nu} - \frac{(x\cdot\nu)|\nabla u|^2}{2} + (x\cdot\nu)F(x,u)\right] d\sigma\\
&+ \int_{B_R} r^k \left[-|\nabla u|^2 - k\frac{(x\cdot\nabla u)^2}{r^2} + \left(\frac{(p-1)(N+k)(1+ar^q) - 2aqr^q}{2(p+1)}\right) u^{p+1}\right] dx,
\end{aligned}$$

where $r = |x|$, $F(x,u) = -\frac{u^2}{2} + \frac{1+a|x|^q}{p+1}u^{p+1}$, and $k = 0$ or k is a real number greater than $-N+1$.

Proof. Set $f(x,u) = -u + (1 + a|x|^q)u^p$. By some calculation and integration by parts, we obtain

$$\begin{aligned}
0 &= \int_{B_R} r^k u(\Delta u + f(x,u))\,dx \qquad (3.2)\\
&= \int_{\partial B_R} r^k u \frac{\partial u}{\partial \nu}\,d\sigma + \int_{B_R} r^k \left[-|\nabla u|^2 - ku(x\cdot\nabla u)r^{-2} + uf(x,u)\right] dx
\end{aligned}$$

and

$$\begin{aligned}
0 = &\int_{B_R} r^k (x\cdot\nabla u)(\Delta u + f(x,u))\,dx \qquad (3.3)\\
= &\int_{\partial B_R} r^k \left[(x\cdot\nabla u)\frac{\partial u}{\partial \nu} - \frac{(x\cdot\nu)|\nabla u|^2}{2} + (x\cdot\nu)F(x,u)\right] d\sigma\\
&+ \int_{B_R} r^k \left[\frac{N-2+k}{2}|\nabla u|^2 - k\frac{(x\cdot\nabla u)^2}{r^2} - (N+k)F(x,u) - x\cdot F_x(x,u)\right] dx,
\end{aligned}$$

where $F(x,u) = \int_0^u f(x,s)\,ds$ and ν is the outer normal of ∂B_R. Multiplying (3.2) by $\frac{N+k}{2}$ and adding it with (3.3), we have proved this lemma. $\square$

Proof of Theorem 1.2. Assume (1.1) has a positive solution u. Our goal is to show this is impossible if a is large. We take $k = -1$ in Lemma 3.1 to obtain

$$\begin{aligned}
&-\int_{B_R} \frac{N-1}{2}\frac{u(x\cdot\nabla u)}{r^3}\,dx \qquad (3.4)\\
= &\int_{\partial B_R} \left[(\frac{x\cdot\nabla u}{r} + \frac{(N-1)u}{2r})\frac{\partial u}{\partial \nu} - \frac{(x\cdot\nu)|\nabla u|^2}{2r} + \frac{(x\cdot\nu)F(x,u)}{r}\right] d\sigma\\
&+ \int_{B_R} \left[-\frac{|\nabla u|^2}{r} + \frac{(x\cdot\nabla u)^2}{r^3} + \frac{1}{(p+1)r}[q_c + (q_c - q)a|x|^q]u^{p+1}\right] dx.
\end{aligned}$$

We note that the integrals on the right side of (3.4) are well defined if $N \geq 3$. Assume $a > 0$. By Theorem 1.1,

$$u(x) = O(|x|^{-\frac{q}{p-1}}) \text{ for } |x| \geq 1. \tag{3.5}$$

By the gradient estimate of elliptic equations, we have

$$\nabla u(x) = O(|x|^{-\frac{q}{p-1}-1}) \text{ for } |x| \geq 1. \tag{3.6}$$

By (3.5), (3.6) and the apriori bound (1.4), the boundary integration in (3.4) can be estimated by $O(R^{-\frac{2q}{p-1}+N-1})$ as $R \to \infty$. Therefore, by $q > q_c$, the decay rate of the boundary integral in (3.4) is

$$O(R^{-\frac{2(q-q_c)}{p-1}}) = o(1) \to 0 \text{ as } R \to \infty. \tag{3.7}$$

We will show that there exists an a_0 such that the right side of (3.4) is negative when R is large and $a \geq a_0$. It is easy to see that

$$-\frac{|\nabla u|^2}{r} + \frac{(x \cdot \nabla u)^2}{r^3} \leq 0. \tag{3.8}$$

We write (1.1) in the form

$$0 = \Delta u + c(x)u, u > 0 \text{ in } \mathbb{R}^N, \tag{3.9}$$

where $c(x) = (1 + a|x|^q)u^{p-1}(x) - 1$. By Theorem 1.1, $c(x)$ is bounded by a constant C_0 independent of u and a. Therefore the Harnack inequality holds and there exists a $c_2 > 0$ such that

$$\max_{|x|\leq 1} u(x) \leq c_2 \min_{|x|\leq 1} u(x). \tag{3.10}$$

We note that for all $a > \frac{q_c}{q-q_c}$, $q_c + (q_c - q)a|x|^q$ changes sign at $|x| = r_a$, where $r_a := [\frac{q_c}{a(q-q_c)}]^{1/q} < 1$. Therefore, we have

$$\begin{aligned}
&\int_{B_1} \frac{1}{r}[q_c + (q_c - q)a|x|^q]u^{p+1}\, dx && (3.11)\\
\leq\ & (q - q_c)a \left[\int_{|x|\leq r_a} \frac{1}{r}[r_a^q - |x|^q](\max_{|x|\leq 1} u)^{p+1}\, dx + \int_{r_a<|x|\leq 1} \frac{1}{r}[r_a^q - |x|^q](\min_{|x|\leq 1} u)^{p+1}\, dx\right]\\
\leq\ & (q - q_c)a\,(\min_{|x|\leq 1} u)^{p+1} \left[\int_{|x|\leq r_a} \frac{1}{r}[r_a^q - |x|^q](c_2)^{p+1}\, dx + \int_{r_a<|x|\leq 1} \frac{1}{r}[r_a^q - |x|^q]\, dx\right].
\end{aligned}$$

Since r_a is decreasing in a and $\lim_{a\to\infty} r_a = 0$, we can choose a large a_0 such that for $a \geq a_0$, the inequality $r_a < 1$ holds and

$$\int_{|x|\leq r_a} \frac{1}{r}[r_a^q - |x|^q](c_2)^{p+1}\, dx + \int_{r_a<|x|\leq 1} \frac{1}{r}[r_a^q - |x|^q]\, dx \leq -\frac{1}{2}\int_{|x|\leq 1} |x|^{q-1}\, dx < 0. \tag{3.12}$$

Therefor when $a \geq a_0$, together with $q_c+(q_c-q)a|x|^q < 0$ for $|x| > 1$, the above inequality implies

$$\int_{B_R} \frac{1}{(p+1)r}\, [q_c + (q_c - q)a|x|^q]\, u^{p+1} dx < 0. \tag{3.13}$$

For $a \geq a_0$, we conclude from (3.7), (3.8) and (3.13) that the right side of (3.4) is negative for large R.

For the term in the left side of (3.4), we have

$$
\begin{aligned}
-\int_{B_R} \frac{u(x\cdot\nabla u)}{r^3}\,dx &= -\int_{B_R} \frac{(x\cdot\nabla u^2)}{2r^3}\,dx \qquad (3.14)\\
&= \begin{cases} \frac{(N-3)}{2}\int_{B_R}\frac{u^2}{r^3}\,dx - \int_{\partial B_R}(x\cdot\nu)\frac{u^2}{2r^3}\,d\sigma, & N\geq 4\\ 2\pi u^2(0) - \int_{\partial B_R}(x\cdot\nu)\frac{u^2}{2r^3}\,d\sigma, & N=3. \end{cases}
\end{aligned}
$$

By (3.5) and $q > q_c$, the boundary terms of the right side of (3.14) tend to zero as $R\to\infty$. Therefore we conclude that $-\frac{N-1}{2}\int_{B_R}\frac{u(x\cdot\nabla u)}{r^3}\,dx$, the left side of (3.4), is positive when R is large, which contradicts the fact that the right side of (3.4) is negative for large R if $a \geq a_0$. The proof of part (a) is complete.

To prove part (b) of the theorem, we assume u is a positive solution of (1.1) on the ball $B_s(0)$ with $s \geq 3$. Assume $R \leq s-2$. We note that by (3.5), (3.6), (3.7) and (3.14), the identity (3.4) can be written as

$$
\begin{aligned}
&\int_{B_R}\left[\frac{|\nabla u|^2}{r} - \frac{(x\cdot\nabla u)^2}{r^3} + \frac{(q-q_c)a}{(p+1)r}[|x|^q - r_a^q]u^{p+1}\right]dx \qquad (3.15)\\
&+ \begin{cases} \frac{(N-1)(N-3)}{4}\int_{B_R}\frac{u^2}{r^3}\,dx, & N\geq 4\\ 2\pi u^2(0), & N=3 \end{cases}\\
=\;& O(R^{-\frac{2(q-q_c)}{p-1}}).
\end{aligned}
$$

From the Harnack inequality (3.10), it follows

$$
u^2(0) \leq c_3\int_{B_1}\frac{u^2}{r^3}\,dx \qquad (3.16)
$$

for some $c_3 > 0$ if $N \geq 4$. Replacing $\min_{|x|\leq 1} u$ by $u(0)$ in (3.13) and (3.12), we have

$$
\begin{aligned}
\int_{|x|\leq R}\frac{1}{r}[|x|^q - r_a^q]u^{p+1}\,dx &\geq \int_{|x|\leq 1}\frac{1}{r}[|x|^q - r_a^q]u^{p+1}\,dx \qquad (3.17)\\
&\geq (c_2u(0))^{p+1}\frac{1}{2}\int_{|x|\leq 1}|x|^{q-1}\,dx.
\end{aligned}
$$

if $a \geq a_0$ and $R \geq 1$. Using (3.16), (3.17) and (3.8), we obtain from (3.15) the estimate

$$
\begin{aligned}
&(q-q_c)au^{p+1}(0) + u^2(0) \qquad (3.18)\\
\leq\;& c_4\int_{B_R}\left[\frac{|\nabla u|^2}{r} - \frac{(x\cdot\nabla u)^2}{r^3} + \frac{(q-q_c)a}{2(p+1)r}[|x|^q - r_a^q]u^{p+1}\right]dx\\
&+ c_4\begin{cases} \frac{(N-1)(N-3)}{4}\int_{B_R}\frac{u^2}{r^3}\,dx, & N\geq 4\\ 2\pi u^2(0), & N=3 \end{cases}\\
=\;& O(R^{-\frac{2(q-q_c)}{p-1}})
\end{aligned}
$$

for some $c_4 > 0$. The proof of part (b) is complete.

$\square$

Proof of Theorem 1.3. Assume (1.1) has a positive solution u on the ball $B_s(0)$ with $s \geq 3$. Let $R \leq s-2$ and $k = 0$ in Lemma 3.1. We obtain the Pohozaev identity

$$
\begin{aligned}
& -\int_{\partial B_R} \left[(x \cdot \nabla u + \frac{N}{2} u) \frac{\partial u}{\partial \nu} - \frac{(x \cdot \nu)|\nabla u|^2}{2} + (x \cdot \nu) F(x,u) \right] d\sigma \\
= & \int_{B_R} \left\{ -|\nabla u|^2 + \frac{1}{p+1} \left[q_\star + (q_\star - q) a r^q \right] u^{p+1} \right\} dx,
\end{aligned}
\tag{3.19}
$$

which has the term $-\int_{B_R} |\nabla u|^2 \, dx$ and is slightly different from the usual form people use.

By (3.5), (3.6) and the apriori bound (1.4), the boundary integration in (3.19) can be estimated by $O(R^{-\frac{2q}{p-1}+N}) = O(R^{-\frac{2(q-q_\star)}{p-1}})$ as $R \to \infty$. Therefore for $q > q_\star$, the left side of (3.19) converges to 0 as $R \to \infty$.

Note that the term $-\int_{B_R} |\nabla u|^2 \, dx$ in (3.19) has a negative sign. Therefore we can proceed as in the proof of Theorem 1.2 to show there exists an a_0 such that if $a \geq a_0$ and $R \geq 1$, the right side of (3.19) is negative and

$$
(q - q_\star) a u^{p+1}(0) = O\left(-\int_{B_R} \left[q_\star + (q_\star - q) a r^q \right] u^{p+1} \, dx \right) = O(R^{-\frac{2(q-q_\star)}{p-1}}),
$$

which implies the inequality in (b). Part (a) follows from (b). The proof is complete. $\square$

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