ALGEBRAIC CONNECTIVITY OF ERDÖS-RÉNYI GRAPHS NEAR THE CONNECTIVITY THRESHOLD

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Abstract. We study the algebraic connectivity λ_2 of an Erdös-Rényi random graph $G(n,p)$ on n vertices, where the edges are chosen with probability $p = p_0 \frac{\log n}{n}$ for some constant p_0. It is well-known that the threshold for connectivity of this graph model is $p_0 = 1$; with high probability the graph is connected when $p_0 > 1$ and disconnected when $p_0 < 1$. In the connected regime where $p_0 > 1$ we show that $\lambda_2 \sim anp$ asymptotically almost surely as $n \to \infty$, where $a \in (0,1)$ satisfies $p_0 = 1 + ap_0(1 - \log a)$. This estimate recovers the two well-known limits $\lambda_2 \to 0$ as $p_0 \to 1^+$ and $\lambda_2 \sim np$ for $np/\log n \to \infty$. We also provide a similar result $\lambda_2(\mathcal{L}) \sim 1$ for the first non-trivial eigenvalue of the normalized graph Laplacian $\mathcal{L}$ in this regime.

Keywords: Erdös-Rényi model, spectral graph theory, graph Laplacian, algebraic connectivity.

1. Introduction. The algebraic connectivity of a graph is defined as the second smallest eigenvalue $\lambda_2(L)$ of the combinatorial graph Laplacian $L = D - A$, where A is its adjacency matrix and D is the diagonal matrix of vertex degrees [20, 15, 30, 9, 16]. This graph invariant arises in the analysis of a variety of graphs processes that describe, for example, information transfer rates for dynamical models [5, 35, 31], robustness and stability in inverse problems [8, 33] and synchronizability in complex networks [3]. It is also widely used in graph partitioning and data clustering algorithms [34] due to its close relationship to the Cheeger constant.

In this paper, we study the algebraic connectivity of Erdös-Rényi graphs $G(n,p)$ on n vertices. In this random graph model, each of the $\binom{n}{2}$ possible edges is included with probability p, independent from every other edge. There is a large literature on Erdös-Rényi graphs [18, 6, 23, 1] and, in particular, their spectral properties [19, 28, 29]. Perhaps the most interesting phenomena related to Erdös-Rényi graphs is the evolution of their connectivity properties as p increases. It is well-known that if $np = p_0 \log n$ for $p_0 < 1$ then the graph is asymptotically almost surely disconnected, while if $p_0 > 1$ then the graph is asymptotically almost surely connected. Thus the graph becomes connected near the *connectivity threshold* $p = \frac{\log n}{n}$ and this transition occurs in the *critical regime* $np = \Theta(\log n)$.

Our goal is to describe the evolution of the algebraic connectivity of an Erdös-Rényi graph in this critical regime. Since the algebraic connectivity of a disconnected graph is zero, we trivially have that $\lambda_2 = 0$ asymptotically almost surely for $p_0 < 1$. The only non-trivial behavior therefore occurs in the connected regime. Numerous previous studies have studied the algebraic connectivity of Erdös-Rényi graphs in the connected case provided np lies above the critical regime. For instance, Juhász [25] showed that

$$\lambda_2 = np + o\left(n^{\frac{1}{2}+\varepsilon}\right) \tag{1.1}$$

for any $\varepsilon > 0$, but this estimate does not capture the behavior of λ_2 for $np = O(n^{\frac{1}{2}})$. Chung, Lu and Vu [11] later improved this estimate in the case of the normalized Laplacian $\mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$ by using the trace method. Their estimates were designed for more general graphs than just $G(n,p)$, but specialized to this case their efforts show

$$\max_{k\geq 2} |\lambda_k(\mathcal{L}) - 1| \leq (1 + o(1))\left(\frac{4}{\sqrt{np}}\right) + \frac{g(n)\log^2 n}{np} \tag{1.2}$$

provided $np/\log^2 n \to \infty$, where $g(n)$ denotes any function tending to infinity arbitrarily slowly. Oliviera [32] and later Chung and Radcliffe [12] improved this result by using a new class of matrix concentration inequalities for random matrices. These results once again apply to more general random graph models, but specialized to $G(n,p)$ yield

$$\max_{k\geq 2} |\lambda_k(\mathcal{L}) - 1| \leq O\left(\sqrt{\frac{\log n}{np}}\right) \tag{1.3}$$

provided $np/\log n \to \infty$, see for instance [12]. While these results apply to the normalized Laplacian $\mathcal{L}$, they extend in a straightforward way to the unnormalized case. Indeed, if $np/\log n \to \infty$ then $G(n,p)$ is essentially regular and D is well-approximated by a diagonal matrix with all entries equal to np, which gives

$$\lambda_2(L) = np + O\left(\sqrt{np\log n}\right). \tag{1.4}$$

A similar estimate also follows directly from the estimates for $\lambda_2(L)$ provided by Coja-Oghlan [13] or by using the results from [19].

In any case, the behavior of $\lambda_2(L)$ is well-understood either below the connectivity threshold $p_0 = 1$ or just above the critical regime. An obvious gap in the literature therefore exists regarding the behavior of λ_2 in the critical regime $np = \Theta(\log n)$. We may therefore naturally ask to what extent the previous estimates such as (1.4) hold in this regime. Very loosely speaking, estimates such as (1.4) rely upon concentration of a "global" graph property, such as the entire adjacency matrix or degree matrix of the graph, around their respective mean. This yields much more precise information, such as simultaneous control over all eigenvalues instead of only λ_2, and the results apply to quite general random graph models as well. However, as (1.4) indicates this concentration breaks down in the critical regime — the order of the error now exactly matches the leading order term. To obtain precise information in the critical regime we cannot rely directly on such concentration results, but instead must sacrifice generality for exactness and compute the quantities of interest directly for $G(n,p)$. As we shall see, the leading order coefficient of λ_2 in (1.1),(1.4) requires modification in the critical regime. Our main result is the following characterization of λ_2:

THEOREM 1.1. *Consider a Erdös-Rényi random graph $G(n,p)$ on n vertices, where the edges are connected with probability*

$$p = p_0 \frac{\log n}{n} \tag{1.5}$$

for $p_0 > 1$ constant in n. Let λ_2 be the first non-zero eigenvalue of the graph Laplacian $L = D - A$. Then

$$\frac{\lambda_2}{pn} \sim a(p_0) + O\left(\frac{1}{\sqrt{np}}\right) \quad \text{as } n \to \infty, \tag{1.6}$$

where $a = a(p_0) \in (0,1)$ denotes the solution to

$$p_0 - 1 = ap_0(1 - \log a). \tag{1.7}$$

More precisely, there exist universal constants C_1, C_2, and C_3 independent of n such that $\left|\frac{\lambda_2}{pn} - a(p_0)\right| \leq \frac{C_1}{\sqrt{np}}$ with the probability at least $1 - C_2 e^{-C_3\sqrt{np}}$.

By pursuing a line of reasoning that mirrors the proof of theorem 1.1, we obtain the following corollary that provides a related estimate for the normalized graph Laplacian.

COROLLARY 1.2. *Let $\lambda_2(\mathcal{L})$ denote the first non-zero eigenvalue of the normalized graph Laplacian $\mathcal{L} = \mathrm{Id} - D^{-1/2}AD^{-1/2}$ and let $\lambda_n(\mathcal{L})$ denote its largest eigenvalue. Under the assumptions of theorem 1.1, the estimates*

$$1 - O\left(\frac{1}{\sqrt{np}}\right) \leq \lambda_2(\mathcal{L}) \leq \frac{1}{1 - O\left(\frac{1}{n}\right)}, \qquad \lambda_n(\mathcal{L}) \leq 1 + O\left(\frac{1}{\sqrt{np}}\right) \tag{1.8}$$

hold with probability at least $1 - C_1 \mathrm{e}^{-C_2\sqrt{np}}$ for C_1, C_2 some positive, universal constants.

REMARK 1.3. *The function $a = a(p_0)$ in Theorem 1.1 has the following limiting properties: $a \to 0^+$ as $p_0 \to 1^+$ and $a \to 1^-$ as $p_0 \to \infty$. Thus, Theorem 1.1 captures the transition of λ_2 between 0 and np as in (1.1). More precisely,*

$$a(p_0) \sim 1 - \sqrt{2/p_0} + O(1/p_0) \text{ for } p_0 \gg 1; \tag{1.9}$$

In Fig. 1.1, we compare the asymptotics (1.9) to $a(p_0)$ as defined in (1.7).

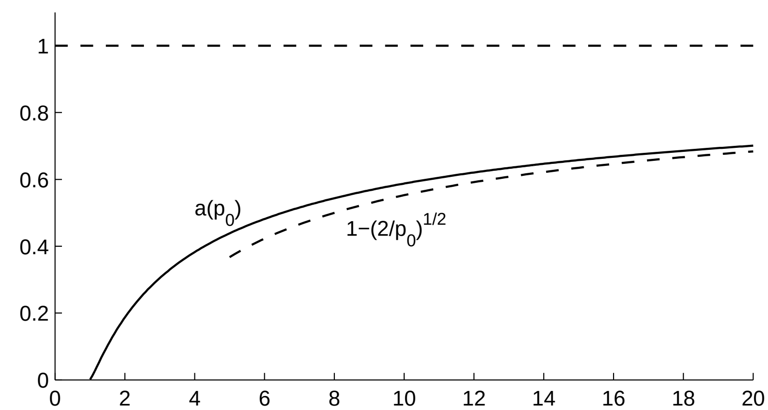

FIG. 1.1. *A comparison of $a(p_0)$ as defined in (1.7) with the asymptotic limit given in (1.9). Note that $a \to 1$ as $p_0 \to \infty$ and $a \to 0^+$ as $p_0 \to 1^+$*

To the best of our knowledge, the result of theorem 1.1 has yet to appear in the literature. Its proof relies on what are, by now, well-established techniques. Our contribution therefore lies in explicitly stating and proving the formula (1.7). A formal derivation of (1.7) is given in Section 2. The proof of Theorem 1.1, given in Section 3, then makes rigorous this formal argument. We may outline the proof as follows. First, following [22] and [37, 38] we demonstrate that, at first order, the eigenvalues of L are the eigenvalues of D; the adjacency matrix, A, can be neglected. The eigenvalues of D are simply the degrees of the graph vertices, which are binomially distributed. The proof is then completed by directly estimating the binomial coefficients to obtain the leading order contribution of the minimal degree, resulting in formula (1.7). We also remark that this formula also appeared in Exercise 3.4 of [6].

Figure 1.2 shows the comparison of the asymptotic formulae, (1.6) and (1.7), with numerical computations of the algebraic connectivity. Although the predicted error of $O(1/\sqrt{\log n})$ is rather large, the agreement is very good even for relatively “small”

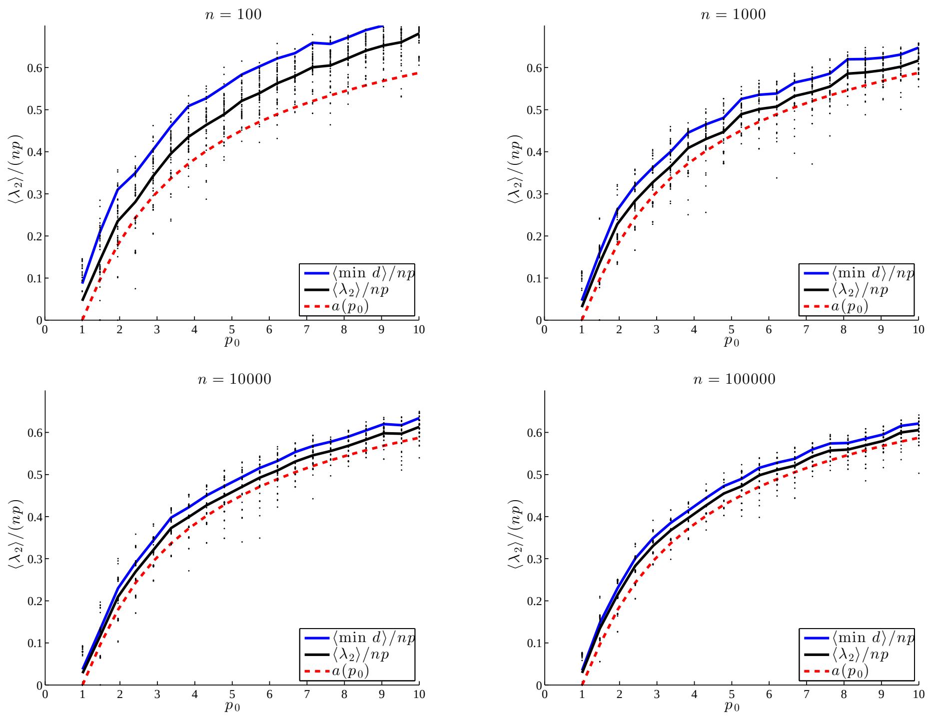

FIG. 1.2. *A comparison of the algebraic connectivity (black), the minimum degree (blue), and the estimate given in Theorem 1.1 (red) of Erdös-Rényi graphs, $G(n,p)$ for four different values of n and $p = p_0 \frac{\log n}{n}$ where $p_0 \in [1,10]$. The black dots represent the algebraic connectivity of 50 sampled graphs and the black curve was obtained by averaging these values. Similarly, the blue curve is the mean of the minimum degrees. As $n \to \infty$, the estimate agrees with the computed values.*

$n = 100$, and gets better slowly with increased n. This approach to the question addressed in Theorem 1.1 is most similar to that used by Jamakovic and Van Mieghem [22]. Their result uses the bound $\lambda_2 \leq \frac{n}{n-1} d_{\min}$ and bounds the minimal degree of the graph using a different analysis than the present work.

We also consider the first non-zero eigenvalue when $p_0 < 1$, i.e. the disconnected regime. In this case, with high probability $G(n,p)$ has isolated vertices, the graph is disconnected, and $\lambda_2 = 0$. The number and size of the connected components is well-studied, (see, for example, [18] and [23, Ch. 5]). As p_0 is decreased below 1, the number of disconnected components (which is the multiplicity of the zero eigenvalue of L) increases. For $1/2 < p_0 < 1$, all such components consist of a single isolated vertex. As p_0 is decreased past $1/2$, some of the disconnected components are a two-vertex tree. As p_0 is decreased past $1/3$, a 3-vertex tree becomes a disconnected component and so on. In their now famous paper, Erdös and Rényi ([18, Thm. 2b]) characterize the number of connected components of each size (see also [23, Thm. 3.30 and Thm 6.38]). Let $p_0 \in (0,1)$ and let N_K denote the number of connected components that have K vertices. If $K > 1/p_0$ then $N_K = 0$ asymptotically almost surely. If $K < 1/p_0$ then

$$N_K \sim \frac{K^{K-2}}{K!} p_0^{K-1} (\log n)^{K-1} n^{1-Kp_0}. \tag{1.10}$$

Additionally, any connected component is asymptotically almost surely a tree (has $K-1$ edges). It then follows that $N_1 \gg N_2 \gg \dots$ with $N_1 \sim n^{1-p_0}$ as $n \to \infty$. Since the dimension of the graph Laplacian kernel is the number of connected components, we obtain

$$\# \text{ of zero eigenvalues of } L \sim \sum_{K=1}^{\lfloor 1/p_0 \rfloor} N_K \sim n^{1-p_0}.$$

In Section 5 we briefly investigate the first non-zero eigenvalue of the graph Laplacian via numerical simulations.

Finally, we mention a few recent studies which are tangentially related to the present one. Much of the previous work in this area relies on well-known techniques, such as the trace method, from random matrix theory [21, 39, 2]. In a similar spirit, [17, 24, 36] establish a law of large numbers for λ_{n-k} as $n \to \infty$. Kahle [26] studied the higher homology groups of the clique complex of Erdös-Rényi graphs. Since the zeroth homology group is simply the number of connected components of the clique complex, these estimates can be considered as higher dimensional analogues of the Erdös-Rényi threshold for connectivity. Bordenave, Caputo, and Chafaï [7] studied the spectrum of *oriented* Erdös-Rényi-type graph Laplacians, which are the infinitesimal generator of a continuous time random walk on the graph. Kolokolnikov [27] studied small graphs with maximal algebraic connectivity subject to constraints on the volume, number of edges, and topology. Zhan et. al. [40] numerically studied a number of questions concerning the eigenvalue distributions of various random graph models, including the Erdös-Rényi model.

Outline.. The outline of the paper is as follows. In Section 2 we give a formal derivation of (1.7). A rigorous proof is given in Section 3. Further discussion and some open questions are given in Section 5.

2. Formal derivation of Theorem 1.1. In this section, we give a formal derivation of (1.7). Recall that the graph Laplacian is given by $L = D - A$ where D is the degree matrix and A is the adjacency matrix. A key observation used in the derivation of our result is the fact that, roughly speaking, the matrix A is of "lower order" than D and can be discarded. The smallest eigenvalue can then be estimated by computing the minimum of the diagonal entries of D so that

$$\lambda_2 \sim \min\left(d_1, \dots d_n\right). \tag{2.1}$$

where $d_i = D_{ii}$ is the degree of the i-th vertex.

Since each degree has distribution Binomial$(n-1, p)$, the probability that $d_i \leq z$ is given by

$$\mathbb{P}(d_i \leq z) = f(z) \qquad \text{where } f(z) := \sum_{i=0}^{\lfloor z \rfloor} \binom{n-1}{i} p^i \left(1-p\right)^{n-1-i}. \tag{2.2}$$

Assuming $i \ll n$, we estimate

$$\begin{aligned} \binom{n-1}{i} p^i \left(1-p\right)^{n-1-i} &\sim \frac{\mu^i e^{-\mu}}{i!}, \quad \text{where } \mu = pn = p_0 \log(n). \\ &\sim \left(2\pi i\right)^{-1/2} \exp(i \log(\mu/i) - \mu + i) \end{aligned}$$

The sum in (2.2) is dominated by its last term if $z < \mu$. To see this, following [4, Ch. 6.7], we estimate the sum in (2.2) by an integral so that

$$f(z) \sim \int_0^z \exp\left(x\log(\mu/x) + x - \mu + \text{l.o.t.}\right)\, dx.$$

Here, l.o.t. denotes terms of lower order. Let $z = a\mu$ and substituting $x = \mu y$, we obtain

$$f(z) \sim \mu \int_0^a \exp\left(\mu(y\log(1/y) + y - 1) + \text{l.o.t.}\right)\, dy$$

Note that $y\log(1/y) + y - 1$ has a maximum at $y = 1$, and is increasing on $[0, 1]$. We are interested in the case where $z < \mu$ implying that the right endpoint $y = a < 1$. Since $\mu \gg 1$, the integrand decays rapidly to the left of $y = a$ and an application of Laplace's method yields

$$\begin{aligned} f(a\mu) &\sim \exp\left(\mu\left\{a\log(1/a) + a - 1\right\} + \text{l.o.t.}\right) \\ &\sim n^{p_0\{a\log(1/a)+a-1\}}\exp(\text{l.o.t.}), \end{aligned} \tag{2.3}$$

where we have used $\mu = p_0 \log n$. Using (2.2) and (2.3) and estimating $(1 - x/n)^n \sim \exp(-x)$ for $x \gg 1$, we obtain

$$\begin{aligned} \mathbb{P}(d_1, \ldots, d_n \geq a\mu) &\sim \prod_i^n \mathbb{P}(d_i \geq a\mu) \\ &\sim (1 - f(a\mu))^n \\ &\sim \exp(-nf(a\mu)) \\ &\sim \exp\left[-n^{p_0\{a\log(1/a)+a-1\}+1}\exp(\text{l.o.t.})\right]. \end{aligned}$$

Define

$$\omega := p_0\left\{a\log(1/a) + a - 1\right\} + 1 \qquad \Longrightarrow \qquad \mathbb{P}(d_1, \ldots, d_n \geq a\mu) \sim \exp\left(-n^{\omega}\right).$$

If $\omega > 0$ then $\mathbb{P}(d_1, \ldots, d_n \geq a\mu) \to 0$ as $n \to \infty$, which together with (2.1), implies that $\lambda_2 > apn$ with high probability. Conversely, if $\omega < 0$ then $\mathbb{P}(d_1, \ldots, d_n \geq a\mu) \to 1$ as $n \to \infty$ implying $\lambda_2 < apn$ with high probability. This shows formally that $\lambda_2 \sim apn$ (as in (1.6)) where a satisfies $\omega = 0$ (as in (1.7)).

3. Proof of Theorem 1.1. Let $A = \{e_{ij}\} \in \mathbb{M}_{n\times n}(\mathbb{R})$ denote the random matrix that corresponds to the adjacency matrix of Erdős-Rényi random graph. That is, $e_{ij} =_d B(1, p)$ where p denotes the probability that an edge exists between two vertices. Define

$$D := \operatorname{diag}(d_1, \ldots, d_n) \qquad d_i := \sum_{j=1}^n e_{ij} =_d B(n, p) \qquad L := D - A$$

and order the eigenvalues of L so that

$$\lambda_n(L) \geq \lambda_{n-1}(L) \geq \lambda_2(L) > \lambda_1(L) = 0.$$

Note that we allow loops in this definition of D and A, but L itself remains unchanged from the usual definition of $G(n,p)$. Let $\mathbf{1}$ denote the constant vector, so that $L\mathbf{1} = \mathbf{0}$ and

$$\lambda_2(L) = \min_{\{\mathbf{v}\perp\mathbf{1}:||\mathbf{v}||=1\}} \langle \mathbf{v}, L\mathbf{v}\rangle.$$

If we denote the corresponding subset of the unit ball $\mathcal{S}^n \subset \mathbb{R}^n$ as

$$\mathcal{S}_0^n := \left\{ \mathbf{v} : \sum_i v_i = 0, \quad \sum_i v_i^2 \leq 1 \right\}, \tag{3.1}$$

then we have the following probabilistic estimate —

THEOREM 3.1. *Let α and p_0 denote arbitrary positive constants. Let A denote the adjacency matrix of a random graph from $G(n,p)$. If $np > p_0 \log n$ then there exists a constant $c = c(\alpha, p_0) > 0$ so that the estimate*

$$\max_{(\mathbf{v},\mathbf{w})\in\mathcal{S}_0^n\times\mathcal{S}^n} |\langle \mathbf{v}, A\mathbf{w}\rangle| \leq c\sqrt{np} \tag{3.2}$$

holds with probability at least $1 - O(n^{-\alpha})$.

This theorem, from [19], allows us to reduce the study of $\lambda_2(L)$ to a study of vertex degrees. Put $np = p_0 \log n$ for some constant $p_0 > 1$ fixed with respect to the number of vertices. Then we have

LEMMA 3.2. *(Reduction to Diagonal) Suppose there exists a $p_0 > 0$ so that $np \geq p_0 \log n$ and $C, c_1 > 0$ so that*

$$|d_{\min} - c_1 np| \leq C\sqrt{np}$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$. Then there exists a $\tilde{C} > 0$ so that

$$|\lambda_2(L) - c_1 np| \leq \tilde{C}\sqrt{np}$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$.

Proof. Note that

$$\lambda_2(L) = \min_{\{\mathbf{v}\perp\mathbf{1}:||\mathbf{v}||=1\}} \langle \mathbf{v}, L\mathbf{v}\rangle = \min_{\{\mathbf{v}\perp\mathbf{1}:||\mathbf{v}||=1\}} \langle \mathbf{v}, D\mathbf{v}\rangle - \langle \mathbf{v}, A\mathbf{v}\rangle.$$

As $\mathbf{v} \in \mathcal{S}_0^n$, the preceeding theorem and the estimate $\lambda_2(L) \leq \frac{n}{n-1} d_{\min}$ imply that there exists a constant $c > 0$ so that

$$\min_{\{\mathbf{v}\perp\mathbf{1}:||\mathbf{v}||=1\}} \langle \mathbf{v}, D\mathbf{v}\rangle - c\sqrt{np} \leq \lambda_2(L) \leq \frac{n}{n-1} d_{\min} \tag{3.3}$$

holds with probability at least $1 - O(n^{-\alpha})$. Clearly it then holds that

$$c_1 np - (C+c)\sqrt{np} \leq d_{\min} - c\sqrt{np} \leq \min_{\{\mathbf{v}\perp\mathbf{1}:||\mathbf{v}||=1\}} \langle \mathbf{v}, D\mathbf{v}\rangle - c\sqrt{np} \leq \lambda_2(L) \leq c_1 np + O(\sqrt{np})$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$, which gives the claim. ☐

The preceeding lemmas allow us to reduce the proof of theorem 1.1 to a verification of the hypothesis

$$|d_{\min} - c_1 np| \leq C\sqrt{np}$$

in Lemma 3.2. To this end, given $X =_d B(n,p)$ and any $0 < c_1 < 1$ define

$$f_n(p, c_1) := \mathbb{P}(X \le c_1 np) = \sum_{i=0}^{i_0} \binom{n}{i} p^i (1-p)^{n-i} \qquad i_0 := \lfloor c_1 np \rfloor \tag{3.4}$$

Then we have the estimate

$$\binom{n}{i_0} p^{i_0}(1-p)^{n-i_0} < f_n(p, c_1) < (c_1 np + 1)\left(\binom{n}{i_0} p^{i_0}(1-p)^{n-i_0}\right). \tag{3.5}$$

The lower bound is obvious since the sum dominates any singleton summand. The upper bound comes from the fact that $i_0 < np$, which implies that

$$\binom{n}{i} p^i (1-p)^{n-i} \le \binom{n}{i_0} p^{i_0}(1-p)^{n-i_0} \qquad (i \le i_0).$$

We may use this simple estimate combined with Stirling's formula to conclude the following lemma, where we envision taking $c_1 = a + o(1)$ for $0 < a < 1$ some constant (in n) fixed.

Lemma 3.3. *Suppose $c_1 = a + o(1)$ for $0 < a < 1$ and that there exist constants $p_0 > 0$ and $C > 0$ so that $p_0 \log n \le np \le C \log n$. Then there exist constants $c', C' > 0$ so that*

$$\frac{c' \mathrm{e}^{np\mathcal{H}(c_1)}}{\sqrt{np}} \le f_n(p, c_1) \le C' \sqrt{np}\, \mathrm{e}^{np\mathcal{H}(c_1)} \tag{3.6}$$

$$\mathcal{H}(c_1) := c_1 + c_1 \log \frac{1}{c_1} - 1. \tag{3.7}$$

Proof. Write

$$\binom{n}{i_0} p^{i_0}(1-p)^{n-i_0} = \mathrm{e}^{\Phi(i_0)}$$
$$\Phi(i_0) := \log n! - \log i_0! - \log(n-i_0)! + i_0 \log p + (n-i_0)\log(1-p).$$

Stirling's formula $\log j! = (j + 1/2)\log j - j + O(1)$ for the factorial then implies that

$$\Phi(i_0) = n \log \frac{n}{n-i_0} - i_0 \log \frac{i_0}{n-i_0} + \frac{1}{2} \log \frac{n}{i_0(n-i_0)} + i_0 \log \frac{p}{1-p} + n\log(1-p) + O(1).$$

Define $\delta_0 \in [c_1 - (np)^{-1}, c_1]$ as $\delta_0 np = i_0$, so that the previous equation simplifies to

$$\Phi(i_0) = -n\log(1-\delta_0 p) + \delta_0 np \log \frac{1-\delta_0 p}{\delta_0(1-p)} - \frac{1}{2}\log np + n\log(1-p) + O(1).$$

The fact that $\log(1-x) = -x + O(x^2)$ as $x \to 0$ then implies

$$\Phi(i_0) = \delta_0 np + \delta_0 np \log \frac{1}{\delta_0} - \frac{1}{2}\log np - np + O(1) + O(np^2).$$

As $\delta_0 = c_1 + O(1/np)$ and $np = O(\log n)$ this relation implies

$$\Phi(i_0) = c_1 np + c_1 np \log \frac{1}{c_1} - \frac{1}{2}\log np - np + O(1) = np\mathcal{H}(c_1) - \frac{1}{2}\log np + O(1),$$

so that

$$\binom{n}{i_0} p^{i_0}(1-p)^{n-i_0} = \mathrm{e}^{\Phi(i_0)} = \frac{\mathrm{e}^{np\mathcal{H}(c_1)}}{\sqrt{np}} \mathrm{e}^{O(1)}.$$

Combining this with (3.5) then gives the claim. □

The preceeding lemma indicates the correct choice of $0 < a < 1$ so that the bound

$$|d_{\min} - anp| \leq C\sqrt{np}$$

could hold asymptotically almost surely. Given $np = p_0 \log n$ for $p_0 > 1$ let $0 < a(p_0) < 1$ denote the solution to

$$p_0 \mathcal{H}(a(p_0)) = -1. \tag{3.8}$$

Formally, for any $\epsilon > 0$ sufficiently small there exists a $\delta > 0$ so that

$$p_0 \mathcal{H}(a(p_0) - \epsilon) \leq -(1+\delta). \tag{3.9}$$

We may therefore conclude that for any such ϵ that

$$\mathbb{P}\left(d_i \leq (a(p_0) - \epsilon)np\right) \leq O(\sqrt{\log n}) n^{-(1+\delta)},$$

and so by the union bound over all diagonal entries

$$\mathbb{P}\left(d_{\min} \leq (a(p_0) - \epsilon)np\right) \leq O(\sqrt{\log n}) n^{-\delta} = o(1).$$

Conversely, for any $\epsilon > 0$ sufficiently small there exists a $\delta > 0$ so that

$$p_0 \mathcal{H}(a(p_0) + \epsilon) \geq -1 + \delta.$$

In this case, if we were to assume independence of the diagonal entries, we can conclude that

$$\begin{aligned}\mathbb{P}\left(d_{\min} \leq (a(p_0)+\epsilon)np\right) &= 1 - \mathbb{P}\left(d_{\min} > (a(p_0)+\epsilon)np\right) = 1 - (1 - f_n(p, a(p_0)+\epsilon))^n \\ &\geq 1 - \left(1 - \frac{c'^{\delta}}{n\sqrt{\log n}}\right)^n \sim 1 - \exp\left(-\frac{c'^{\delta}}{\sqrt{\log n}}\right) = 1 - o(1).\end{aligned}$$

In other words, we formally have $(a(p_0) - \epsilon)np \leq d_{\min} \leq (a(p_0)+\epsilon)np$ asymptotically almost surely. To remove the "formally" and to establish the precise error bound

$$|d_{\min} - a(p_0)np| \leq C\sqrt{np}$$

we must choose ϵ, δ carefully and resolve the issue that the d_i are not, in fact, independent. The following lemma makes the foregoing precise, and completes the proof of Theorem 1.1 by fulfilling the hypothesis of Lemma 3.2.

Lemma 3.4. *(Minimum Degree) Let $np = p_0 \log n$ for $p_0 > 1$ constant with respect to n and $a(p_0)$ denote the solution to*

$$p_0 - 1 = a p_0 (1 - \log a). \tag{3.10}$$

satisfying $0 < a(p_0) < 1$. Then

$$|d_{\min} - a(p_0)np| \leq C\sqrt{np}$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$.

Proof. Let

$$c_1^{\pm} := a(p_0) \pm \frac{1}{\sqrt{np}}.$$

Then by the mean value theorem, there exist $a_+ \in (a(p_0), a(p_0) + 1/\sqrt{np})$ and $a_- \in (a(p_0) - 1/\sqrt{np}), a(p_0))$ so that

$$\mathcal{H}(c_1^{\pm}) = \mathcal{H}(a(p_0)) \pm \frac{\mathcal{H}'(a_{\pm})}{\sqrt{np}} \qquad \mathcal{H}'(a_{\pm}) = \mathcal{H}'(a(p_0)) + o(1) > 0 \tag{3.11}$$

for all n sufficiently large. This implies that for each $1 \le i \le n$ that

$$\mathbb{P}(d_i \le a(p_0)np - \sqrt{np}) \le O(\sqrt{np})n^{-1}\mathrm{e}^{-\sqrt{np}\mathcal{H}'(a_-)}, \tag{3.12}$$

and so by the union bound that

$$\mathbb{P}\left(d_{\min} \le a(p_0)np - \sqrt{np}\right) \le O(\sqrt{np})\mathrm{e}^{-\sqrt{np}\mathcal{H}'(a_-)} = O(\mathrm{e}^{-\Omega(\sqrt{np})}). \tag{3.13}$$

In other words,

$$d_{\min} \ge a(p_0)np - \sqrt{np} \tag{3.14}$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$. The converse direction proves slightly more difficult due to the fact that the diagonal entries d_i exhibit a mild dependence that results from the undirected graph. Let $X_i := \mathbf{1}_{\{d_i \le c_1^+ np\}}$ denote the indicator of the event that the i^{th} diagonal satisfies $d_i \le a(p_0)np + \sqrt{np}$ and define

$$N_0 := \sum_{i=1}^{n} X_i$$

as the total number of such events that occur. Let $\mu_0 := \mathbb{E}(N_0) = nf_n(p, c_1^+)$ denote the expected number of such entries. Chebyshev's inequality then implies that (writing f_n as shorthand for $f_n(p, c_1^+)$) for any $\gamma > 0$ the inequality

$$\mathbb{P}(|N_0 - \mu_0| > \gamma n f_n) \le \frac{\mathrm{Var}(N_0)}{\gamma^2 n^2 f_n^2} \tag{3.15}$$

holds. The variance satisfies

$$\mathrm{Var}(N_0) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i=1}^{n}\sum_{j>i} \mathrm{Cov}(X_i, X_j) = nf_n(1 - f_n) + 2\sum_{i=1}^{n}\sum_{j>i} \mathrm{Cov}(X_i, X_j),$$

whereas the covariance satisfies

$$\mathrm{Cov}(X_i, X_j) = \mathbb{P}(X_i = 1 \cap X_j = 1) - f_n^2.$$

Recalling the definition of D shows that the vertex degrees d_i and d_j decompose as

$$d_i = \sum_{k=1}^{n} e_{ik} \qquad d_j = \sum_{k=1}^{n} e_{jk},$$

which obviates the fact that the only dependence between d_i and d_j occurs via the entry e_{ij}; indeed, the entries $\{e_{ik}\}_{k\neq j}^n, \{e_{jk}\}_{k\neq i}^n$ are independent. With this in mind, define

$$\tilde{d}_i := \sum_{k\neq j}^n e_{ik} = d_i - e_{ij}$$

and define $\tilde{d}_j$ similarly. Conditioning on the possible values of $e_{ij} \in \{0,1\}$ shows

$$\mathbb{P}(X_i = 1 \cap X_j = 1) = \mathbb{P}(\tilde{d}_i + 1 \leq c_1^+ np)\mathbb{P}(\tilde{d}_j + 1 \leq c_1^+ np)p + \mathbb{P}(\tilde{d}_i \leq c_1^+ np)\mathbb{P}(\tilde{d}_j \leq c_1^+ np)(1-p).$$

Note that

$$\mathbb{P}(\tilde{d}_i \leq c_1^+ np) = \sum_{i=0}^{\lfloor c_1^+ np \rfloor} \binom{n-1}{i} p^i (1-p)^{n-(i+1)},$$

and also that

$$\binom{n-1}{i} p^i (1-p)^{n-(i+1)} \leq \left(1 + \frac{p}{1-p}\right) \binom{n}{i} p^i (1-p)^{n-i}.$$

These facts imply

$$\mathbb{P}(\tilde{d}_i \leq c_1^+ np) \leq \left(1 + \frac{p}{1-p}\right) f_n.$$

The fact that $\{\tilde{d}_i + 1 \leq c_1^+ np\} \subset \{X_i = 1\}$ implies

$$\mathbb{P}(\tilde{d}_i + 1 \leq c_1^+ np) \leq f_n,$$

which yields as a consequence the estimate

$$\mathbb{P}(X_i = 1 \cap X_j = 1) \leq f_n^2 + O(1) f_n^2 p.$$

Substituting this estimate into the covariance shows that

$$\mathrm{Var}(N_0) \leq n f_n + O(1) n^2 f_n^2 p.$$

This estimate combines with (3.15) to show that for any fixed $\gamma > 0$ the inequality

$$\mathbb{P}(|N_0 - \mu_0| > \gamma n f_n) \leq \frac{1}{\gamma^2 n f_n} + \frac{O(1)p}{\gamma^2}. \tag{3.16}$$

holds. Moreover, the previous lemma yields that

$$n f_n \geq c' \frac{\mathrm{e}^{\mathcal{H}'(a_+)\sqrt{np}}}{\sqrt{np}} \to \infty,$$

so that by taking $\gamma = 1/2$ (for instance), it follows that $N_0 \geq n f_n/2$ with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$. Thus the random graph has at least $n f_n/2 \to \infty$ vertices that satisfy

$$d_i \leq a(p_0)np + \sqrt{np},$$

with at least this probability, so in particular

$$d_{\min} \leq a(p_0)np + \sqrt{np},$$

with at least this probability as well. Combining this with the previous case shows that

$$|d_{\min} - a(p_0)np| \leq \sqrt{np}$$

with probability at least $1 - O(\mathrm{e}^{-\Omega(\sqrt{np})})$, as claimed. ∎

4. Proof of Corollary 1.2. Just as results for the normalized Laplacian $\mathcal{L} = \mathrm{Id} - D^{-1/2}AD^{-1/2}$ furnish results for the unnormalized case $L = D - A$ when $G(n,p)$ is regular, lemma 3.4 regarding the minimal degree of $G(n,p)$ also combines with theorem 3.1 to provide spectral estimates for the normalized graph Laplacian just as easily as the unnormalized case. The following lemmas suffice to establish the corollary, which yields a result similar in spirit to those presented in [11, 13, 10, 14, 32, 12, 29].

LEMMA 4.1. *Let $A = \{e_{ij}\}$ denote the adjacency matrix of a random graph drawn from $G(n,p)$ and define*

$$X := \sum_{i,j=1}^{n} e_{ij} = \sum_{i=1}^{n} d_i. \tag{4.1}$$

If there exists a $c_0 > 0$ so that $np > c_0 \log n$ then $X = n^2p + O(n\sqrt{np})$ with probability at least $1 - O(\mathrm{e}^{-n/4})$.

Proof. Let $Y := \sum_{i=1}^{n}\sum_{j=i+1}^{n} e_{ij}$ and note that $X = 2Y$ due to symmetry. Moreover, Y is a sum of $n(n-1)/2$ i.i.d. Bernoulli random variables. For such sums, the Chernoff bound states

$$\mathbb{P}\left(|Y - \mathbb{E}(Y)| \geq \lambda\sigma\right) \leq 2\mathrm{e}^{-\frac{\lambda^2}{4}}$$

provided $\mathrm{Var}(X) \leq \sigma^2$ and $\lambda \leq 2\sigma$. As Y is binomial, $\mathbb{E}(Y) = n(n-1)p/2$ and $\mathrm{Var}(Y) = n(n-1)p(1-p)/2 \leq n^2p$. Choosing $\sigma = n\sqrt{p}$ and $\lambda = \sqrt{n}$ in the Chernoff bound then implies that

$$\mathbb{P}\left(|Y - n(n-1)p/2| \geq n\sqrt{np}\right) \leq 2\mathrm{e}^{-n/4}.$$

Thus $X = 2Y = n^2p + O(n\sqrt{np})$ with probability at least $1 - O(\mathrm{e}^{-n/4})$, which gives the result. ∎

LEMMA 4.2. *Let $d_{\min}$ denote the minimum degree of an Erdös-Rényi random graph $G(n,p)$. Let $np = p_0 \log n$ for $p_0 > 1$ and suppose*

$$d_{\min} \geq a(p_0)np + O(\sqrt{np}).$$

Then

$$1 - O\left(\frac{1}{\sqrt{np}}\right) \leq \lambda_2(\mathcal{L}) \leq \frac{1}{1 - O\left(\frac{1}{n}\right)}, \qquad \lambda_n(\mathcal{L}) \leq 1 + O\left(\frac{1}{\sqrt{np}}\right). \tag{4.2}$$

Proof. Note that

$$\lambda_2(\mathcal{L}) = \min_{\{\mathbf{v} \perp D^{1/2}\mathbf{1} : ||\mathbf{v}||=1\}} \langle \mathbf{v}, \mathcal{L}\mathbf{v} \rangle.$$

Let $\mathbf{v}_2$ denote any normalized ($||\mathbf{v}_2|| = 1$) eigenvector satisfying $\mathcal{L}\mathbf{v}_2 = \lambda_2(\mathcal{L})\mathbf{v}_2$ and put

$$D^{-1/2}\mathbf{v}_2 = \alpha\mathbf{1} + \mathbf{1}^{\perp},$$

where α is the mean of $D^{-1/2}\mathbf{v}_2$ and $\mathbf{1}^{\perp}$ denotes its orthogonal projection onto mean-zero vectors. Now note that the fact $\langle \mathbf{v}_2, D^{1/2}\mathbf{1}\rangle = 0$ implies

$$\begin{aligned}\lambda_2(\mathcal{L}) &= 1 - \langle D^{-1/2}\mathbf{v}_2, AD^{-1/2}\mathbf{v}_2\rangle = 1 - \langle D^{-1/2}\mathbf{v}_2, \alpha A\mathbf{1}\rangle - \langle D^{-1/2}\mathbf{v}_2, A\mathbf{1}^{\perp}\rangle \\ &= 1 - \langle D^{-1/2}\mathbf{v}_2, \alpha D\mathbf{1}\rangle - \langle D^{-1/2}\mathbf{v}_2, A\mathbf{1}^{\perp}\rangle = 1 - \langle D^{-1/2}\mathbf{v}_2, A\mathbf{1}^{\perp}\rangle \\ &\geq 1 - O(\sqrt{np})||D^{-1/2}\mathbf{v}_2||\,||\mathbf{1}^{\perp}||,\end{aligned}$$

where the last inequality follows from theorem 3.1. But note that $||\mathbf{1}^{\perp}|| \leq ||D^{-1/2}\mathbf{v}_2||$ since the decomposition of $D^{-1/2}\mathbf{v}_2$ is orthogonal. Thus the minimum degree assumption implies

$$\begin{aligned}\lambda_2(\mathcal{L}) &\geq 1 - O(\sqrt{np})||D^{-1/2}\mathbf{v}_2||^2 = 1 - O(\sqrt{np})\sum_{i=1}^{n}\frac{(\mathbf{v}_2)_i^2}{d_i} \\ &\geq 1 - O\left(\frac{\sqrt{np}}{a(p_0)np + O(\sqrt{np})}\right)||\mathbf{v}_2||^2 = 1 - O\left(\frac{1}{\sqrt{np}}\right).\end{aligned}$$

To see that the reverse inequality also holds, consider the test vector

$$\mathbf{w} := D^{1/2}\mathbf{e}_{\min} - \langle D^{1/2}\mathbf{e}_{\min}, D^{1/2}\mathbf{1}\rangle\frac{D^{1/2}\mathbf{1}}{||D^{1/2}\mathbf{1}||^2}$$

where $\mathbf{e}_{\min}$ denotes the standard basis vector corresponding to the vertex of minimal degree. By definition, since $\langle \mathbf{w}, D^{1/2}\mathbf{1}\rangle = 0$ the inequality

$$\lambda_2(\mathcal{L})||\mathbf{w}||^2 \leq \langle \mathbf{w}, \mathcal{L}\mathbf{w}\rangle = d_{\min}$$

holds. Under the minimal degree assumption, the norm $||\mathbf{w}||$ satisfies

$$\begin{aligned}||\mathbf{w}||^2 &= d_{\min} - \frac{\langle D^{1/2}\mathbf{e}_{\min}, D^{1/2}\mathbf{1}\rangle^2}{||D^{1/2}\mathbf{1}||^2} = d_{\min} - \frac{d_{\min}^2}{\sum_{i=1}^{n} d_i} \\ &= d_{\min}\left(1 - \frac{a(p_0)np + O(\sqrt{np})}{n^2p + O(n\sqrt{np})}\right) = d_{\min}\left(1 - \frac{a(p_0) + O(1/\sqrt{np})}{n(1 + O(1/\sqrt{np}))}\right).\end{aligned}$$

The estimate on $\sum_{i=1}^{n} d_i$ follows by applying the previous lemma on the sum of vertex degrees. All together, this yields the claimed estimate

$$1 - O\left(\frac{1}{\sqrt{np}}\right) \leq \lambda_2(\mathcal{L}) \leq \frac{1}{1 - \frac{a(p_0) + O(1/\sqrt{np})}{n(1 + O(1/\sqrt{np}))}} = \frac{1}{1 - O\left(\frac{1}{n}\right)}$$

for $\lambda_2(\mathcal{L})$.

The estimate for $\lambda_n(\mathcal{L})$ follows in a similar way. Let $\mathbf{v}_n$ denote a normalized ($||\mathbf{v}_n|| = 1$) eigenvector $\mathcal{L}\mathbf{v}_n = \lambda_n\mathbf{v}_n$, and decompose $D^{-1/2}\mathbf{v}_n$ into its mean and mean-zero components

$$D^{-1/2}\mathbf{v}_n = \alpha\mathbf{1} + \mathbf{1}^{\perp},$$

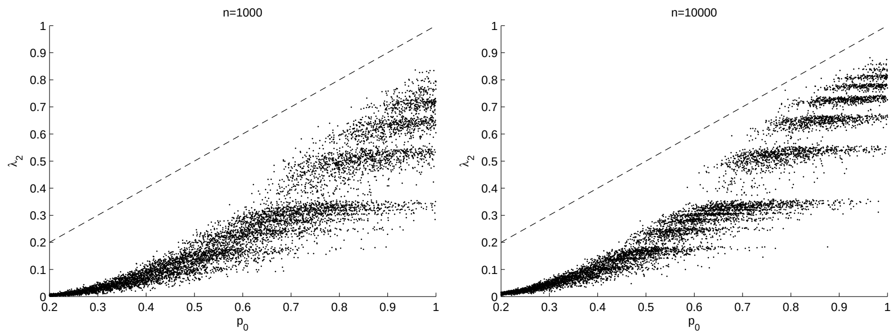

FIG. 5.1. *Density distribution of the algebraic connectivity of the giant component in the regime $p_0 < 1$.*

exactly as before. The fact that $\langle \mathbf{v}_n, D^{1/2}\mathbf{1}\rangle = 0$ then combines with theorem 3.1 to imply

$$\begin{aligned}\lambda_n(\mathcal{L}) &= 1 - \langle D^{-1/2}\mathbf{v}_n, AD^{-1/2}\mathbf{v}_n\rangle = 1 - \langle D^{-1/2}\mathbf{v}_n, \alpha A\mathbf{1}\rangle - \langle D^{-1/2}\mathbf{v}_n, A\mathbf{1}^{\perp}\rangle \\ &= 1 - \langle D^{-1/2}\mathbf{v}_n, \alpha D\mathbf{1}\rangle - \langle D^{-1/2}\mathbf{v}_n, A\mathbf{1}^{\perp}\rangle = 1 - \langle D^{-1/2}\mathbf{v}_n, A\mathbf{1}^{\perp}\rangle \\ &\le 1 + O(\sqrt{np})||D^{-1/2}\mathbf{v}_n||\,||\mathbf{1}^{\perp}|| \le 1 + O(\sqrt{np})||D^{-1/2}\mathbf{v}_n||^2.\end{aligned}$$

That

$$O(\sqrt{np})||D^{-1/2}\mathbf{v}_n||^2 = O\left(\frac{1}{\sqrt{np}}\right)$$

then follows by using the same argument used for the second eigenvector. The conclusion of corollary 1.2 then follows from the estimate provided for the minimal degree. ∎

5. Discussion. In this paper, we gave a precise asymptotic description of the evolution of algebraic connectivity of a random graph in the critical regime $p = p_0\frac{\ln n}{n}$ above the connectivity threshold $p_0 = 1$. Our description captures the transition of λ_2 between 0 (as $p_0 \to 1^+$) and np (as p_0 becomes large). For $p_0 > 1$, we showed that the distribution of λ_2, as well as that of the minimum degree, approaches a delta function centered at $a(p_0)$ given by (1.7), as $n \to \infty$. On the other hand, for $0 < p_0 < 1$, the graph is disconnected almost surely and its algebraic connectivity is zero. We gave a simple estimate of n^{1-p_0} for the number of zero eigenvalues in this case, which corresponds to the number of disconnected components.

Since $\lambda_2(G) = 0$ for $p_0 < 1$, a natural question to ask then is: what is the distribution of the smallest *non-zero* eigenvalue of the Laplacian? For $p_0 \in (0, 1)$, this corresponds to $\lambda_2(G_1)$ where G_1 is the largest connected component of G (*i.e.* the so-called giant component). In Figure 5.1, we plot $\lambda_2(G_1)$ for 10,000 simulations with $n = 1,000$ and $n = 10,000$ as p_0 is increased from 0.2 to 1. When $p_0 < 1$, the minimum degree of G_1 is exactly 1, so it follows that $\lambda_2(G_1) \le 1$. Tighter bounds than this are elusive because of the high complexity ($|E| - |V| + 1$) of the giant component [18]. On the other hand, numerical simulations suggest

$$\lambda_2(G_1) < p_0,$$

as indicated by the dashed line in Figure 5.1. Additionally, Figure 5.1 shows that for fixed $p_0 < 1$, (*e.g.* $p_0 = 0.7$) the distribution of $\lambda_2(G_1)$ does not appear to concentrate at any single value as $n \to \infty$. The distribution appears to have a band-like structure with gaps, which persists as n is increased. It would be very interesting to study the distribution of these bands.

For the normalized Laplacian $\mathcal{L}$ Corrollary 1.2 shows that its eigenvalues concentrate near 1 in the regime $p = O(\log(N)/N)$. However the convergence is very slow (of $O(\sqrt{1/\log(N)})$). In practical terms, this means that λ_2 of $\mathcal{L}$ can deviate very significantly from 1 even for very large N. To illustrate this, we took $p_0 = 2$ and computed the average of λ_2 and λ_n for several values of N resulting in the following table:

N	$\lambda_2(\mathcal{L})$	$\lambda_n(\mathcal{L})$
100	0.4115 (0.014)	1.603 (0.013)
500	0.4655 (0.004)	1.538 (0.0048)
1000	0.4863 (0.0029)	1.515 (0.0026)
5000	0.5312 (0.00089)	1.469 (0.00082)
10000	0.5475 (0.00068)	1.4524 (0.00028)

(the value in brackets denote the standard deviation of 100 simulations). Computing the next term in the expansion is an open problem.

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REFERENCES

[1] N. Alon and J. H. Spencer. *The probabilistic method.* Wiley, 2004.

[2] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices.* Cambridge University Press, 2010.

[3] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou. Synchronization in complex networks. *Physics Reports*, 469:93–153, 2008.

[4] Carl M Bender and Steven A Orszag. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*, volume 1. Springer, 1999.

[5] A. Björner, L. Lovász, and P. W. Shor. Chip-firing games on graphs. *European J. Combin*, 12(4), 1991.

[6] B. Bollobás. *Modern graph theory.* Springer, 1998.

[7] C. Bordenave, P. Caputo, and D. Chafai. Spectrum of Markov generators on sparse random graphs. *arXiv preprint arXiv:1202.0644*, 2012.

[8] N. Boumal, A. Singer, P.-A. Absil, and V. D. Blondel. Cramér-Rao bounds for synchronization of rotations. *arXiv preprint arXiv:1211.1621*, 2012.

[9] F. R. K. Chung. *Spectral Graph Theory.* AMS, 1997.

[10] F. R. K. Chung and L. Lu. *Complex graphs and networks.* AMS, 2006.

[11] F. R. K. Chung, L. Lu, and V. H. Vu. The spectra of random graphs with given expected degrees. *Internet Mathematics*, 1:257–275, 2003.

[12] F. R. K. Chung and M. Radcliffe. On the spectra of general random graphs. *The electronic journal of combinatorics*, 18(P215), 2011.

[13] A. Coja-Oghlan. On the Laplacian eigenvalues of $G_{n,p}$. *Combinatorics, Probability, & Computing*, 16(6):923–946, 2007.

[14] A. Coja-Oghlan and A. Lanka. The spectral gap of random graphs with given expected degrees. *the electronic journal of combinatorics*, 16(R138):1, 2009.

[15] D. M. Cvetkovic, M. Doob, and H. Sachs. *Spectra of graphs: theory and application*, volume 413. Academic press New York, 1980.

[16] N. M. M. de Abreu. Old and new results on algebraic connectivity of graphs. *Linear algebra and its applications*, 423(1):53–73, 2007.

[17] X. Ding and T. Jiang. Spectral distributions of adjacency and Laplacian matrices of random graphs. *arXiv preprint arXiv:1011.2608*, 2010.

[18] P. Erdös and A A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci*, 5:17–61, 1960.

[19] U. Feige and E. Ofek. Spectral techniques applied to sparse random graphs. *Random Structures & Algorithms*, 27(2):251–275, 2005.

[20] M. Fiedler. Algebraic connectivity of graphs. *Czech. Math. J.*, 23:298–305, 1973.

[21] Z. Füredi and K. János. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[22] A. Jamakovic and P. Van Mieghem. On the robustness of complex networks by using the algebraic connectivity. In *Networking 2008 Ad Hoc and Sensor Networks, Wireless Networks, Next Generation Internet*, pages 183–194. Springer, 2008.

[23] S. Janson, T. Luczak, and A. Rucinski. *Random graphs*. Wiley, 2000.

[24] T. Jiang. Low eigenvalues of Laplacian matrices of large random graphs. *Probability Theory and Related Fields*, 153(3-4):671–690, 2012.

[25] F. Juhász. The asymptotic behaviour of Fiedler's algebraic connectivity for random graphs. *Discrete mathematics*, 96(1):59–63, 1991.

[26] M. Kahle. Topology of random clique complexes. *Discrete Mathematics*, 309(6):1658–1671, 2009.

[27] T. Kolokolnikov. Maximizing algebraic connectivity for certain families of graphs. *submitted*, 2013.

[28] Michael Krivelevich and Benny Sudakov. The largest eigenvalue of sparse random graphs. *Combinatorics, Probability and Computing*, 12(01):61–72, 2003.

[29] L. Lu and X. Peng. Spectra of edge-independent random graphs. *arXiv preprint arXiv:1204.6207*, 2012.

[30] B. Mohar. The Laplacian spectrum of graphs. In *Graph Theory, Combinatorics, and Applications*, volume 2, pages 871–898. Wiley, 1991.

[31] R. Olfati-Saber, A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[32] R. I. Oliveira. Concentration of the adjacency matrix and of the laplacian in random graphs with independent edges. *arXiv preprint arXiv:0911.0600v2*, 2010.

[33] B. Osting, C. Brune, and S. Osher. Enhanced statistical rankings via targeted data collection. *JMLR, W&CP*, 28(1):489–497, 2013.

[34] J. Shi and J. Malik. Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(8):888–905, 2000.

[35] J. Sun, S. Boyd, L. Xiao, and P. Diaconis. The fastest mixing Markov process on a graph and a connection to a maximum variance unfolding problem. *SIAM Review*, 48:2006, 2004.

[36] L. V. Tran, V. H. Vu, and K. Wang. Sparse random graphs: Eigenvalues and eigenvectors. *Random Structures & Algorithms*, 42(1):110–134, 2013.

[37] James von Brecht, Theodore Kolokolnikov, Andrea L Bertozzi, and Hui Sun. Swarming on random graphs. *Journal of Statistical Physics*, pages 1–24, 2013.

[38] James von Brecht, Benny Sudakov, and Andrea L. Bertozzi. Swarming on random graphs II. *Submitted*, 2013.

[39] V. H. Vu. Spectral norm of random matrices. *Combinatorica*, 27(6):721–736, 2007.

[40] C. Zhan, G. Chen, and L. F. Yeung. On the distributions of Laplacian eigenvalues versus node degrees in complex networks. *Physica A: Statistical Mechanics and its Applications*, 389(8):1779–1788, 2010.