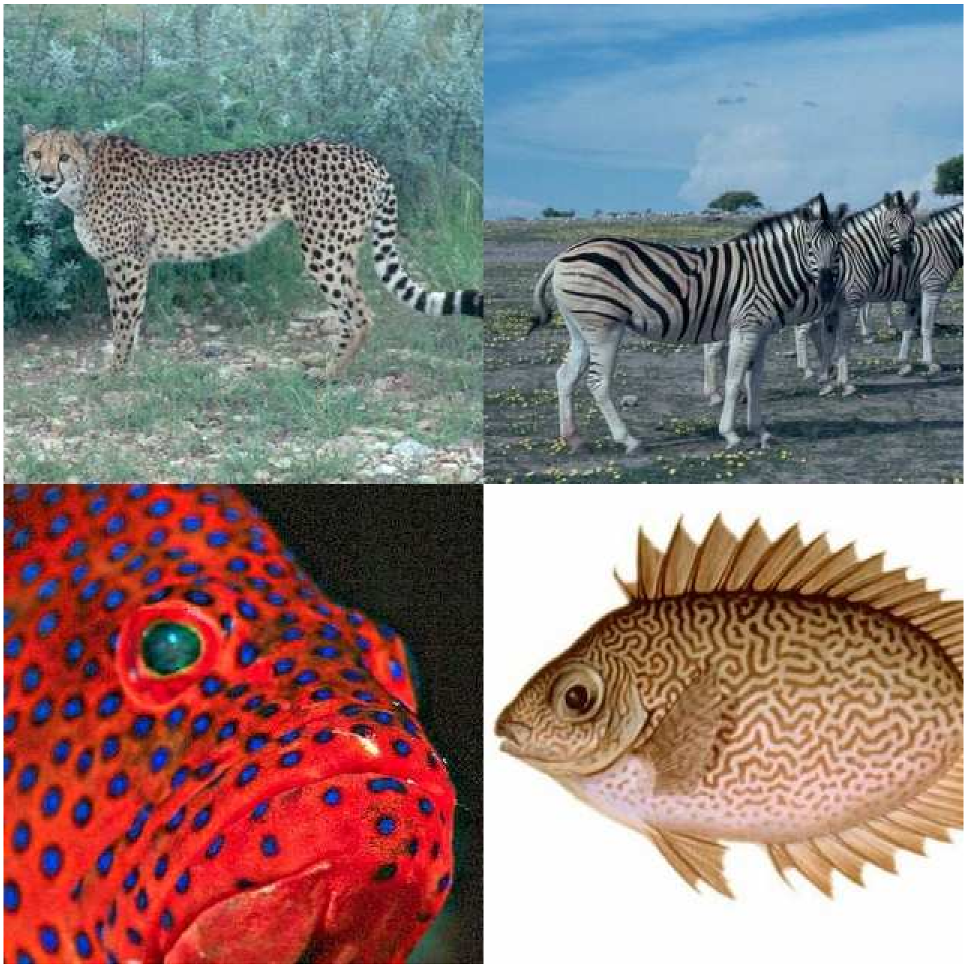
Spots, stripes, and labyrinths in reaction diffusion systems

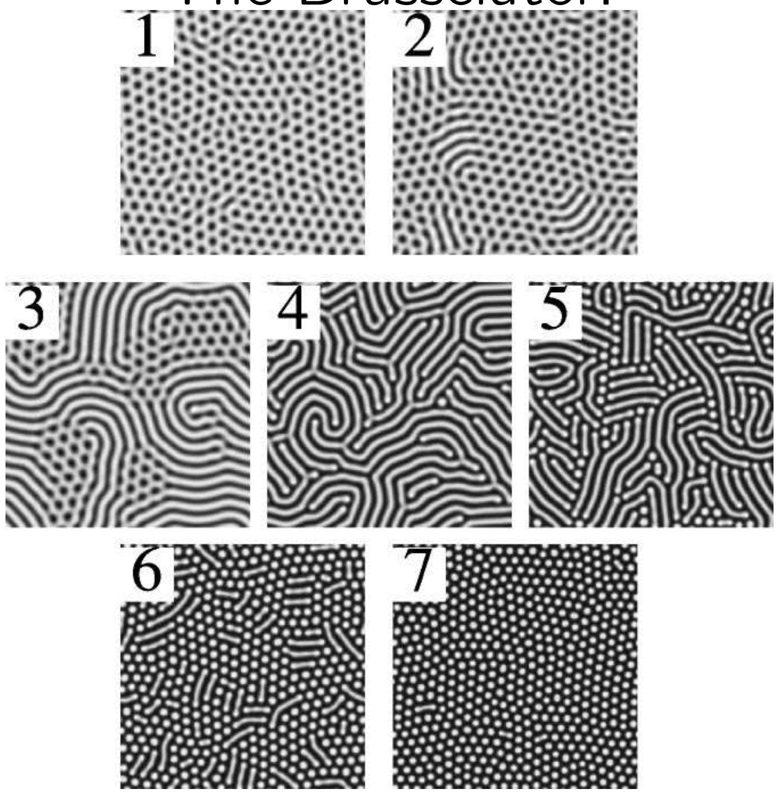


Theodore Kolokolnikov

Joint work with
Michael Ward and Juncheng Wei

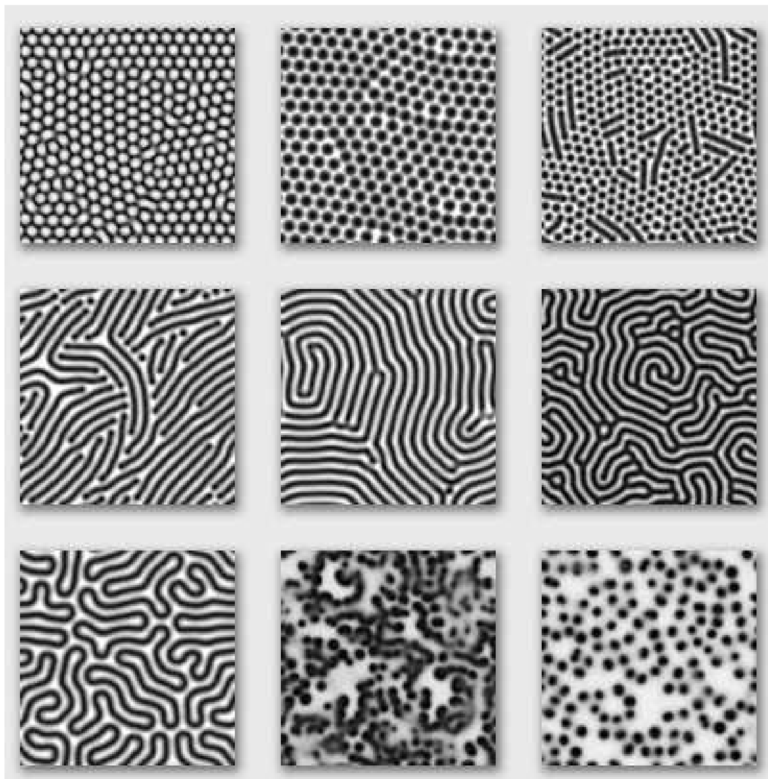
Some reaction diffusion patterns in 2D

The Brusselator:



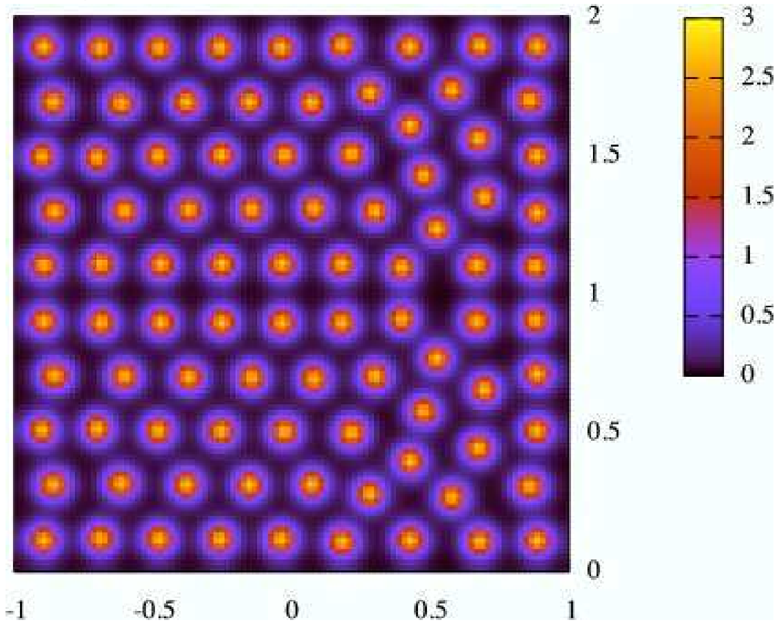
Reference: B. Peña and C. Pérez-García, *Stability of Turing patterns in the Brusselator model*, Phys. Rev. E. Vol. 64(5), 2001.

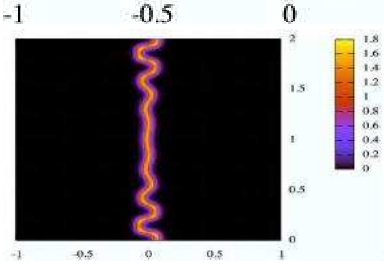
Gray Scott Model:

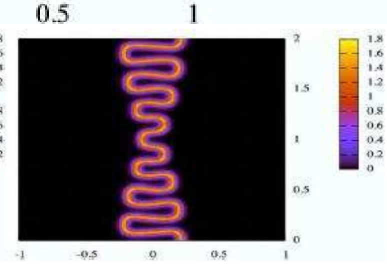


Reference: the Xmorphia website

Gierer-Meinhardt model:

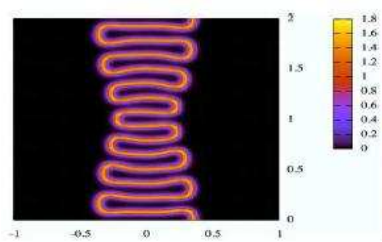


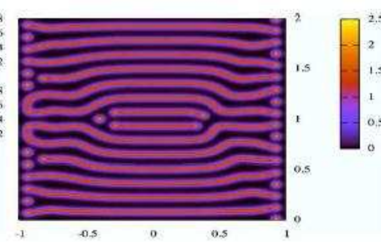




(a) Experiment 5: $t = 1400$

(b) Experiment 5: $t = 1600$

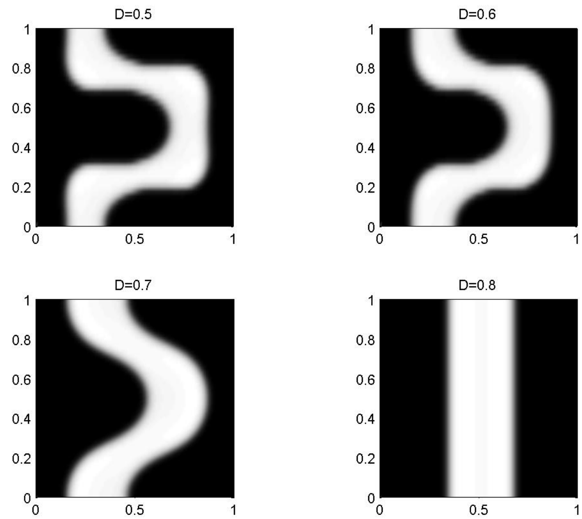


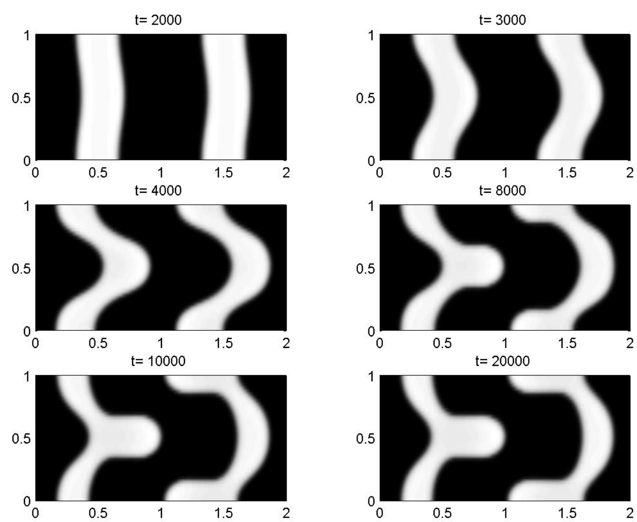


(c) Experiment 5: $t = 1800$

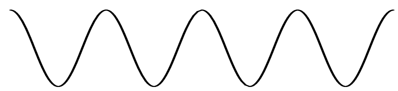
(d) Experiment 5: $t = 4300$

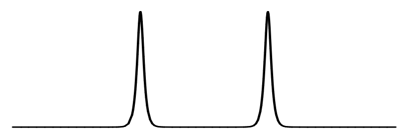
Gierer-Meinhardt model with large saturation:



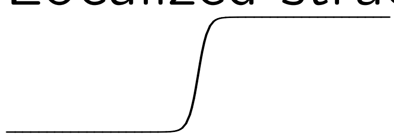
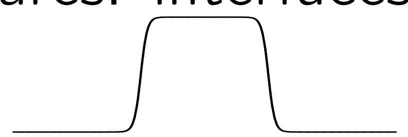


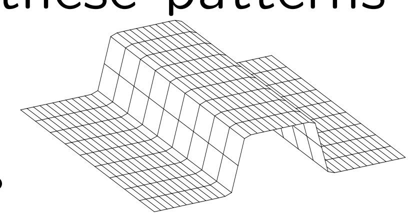
Pattern types

- Turing patterns 

- Localized structures: spikes 

- Localized structures: interfaces, mesas

- What is the stability when these patterns are extended trivially to 2-D? 

Gray Scott model

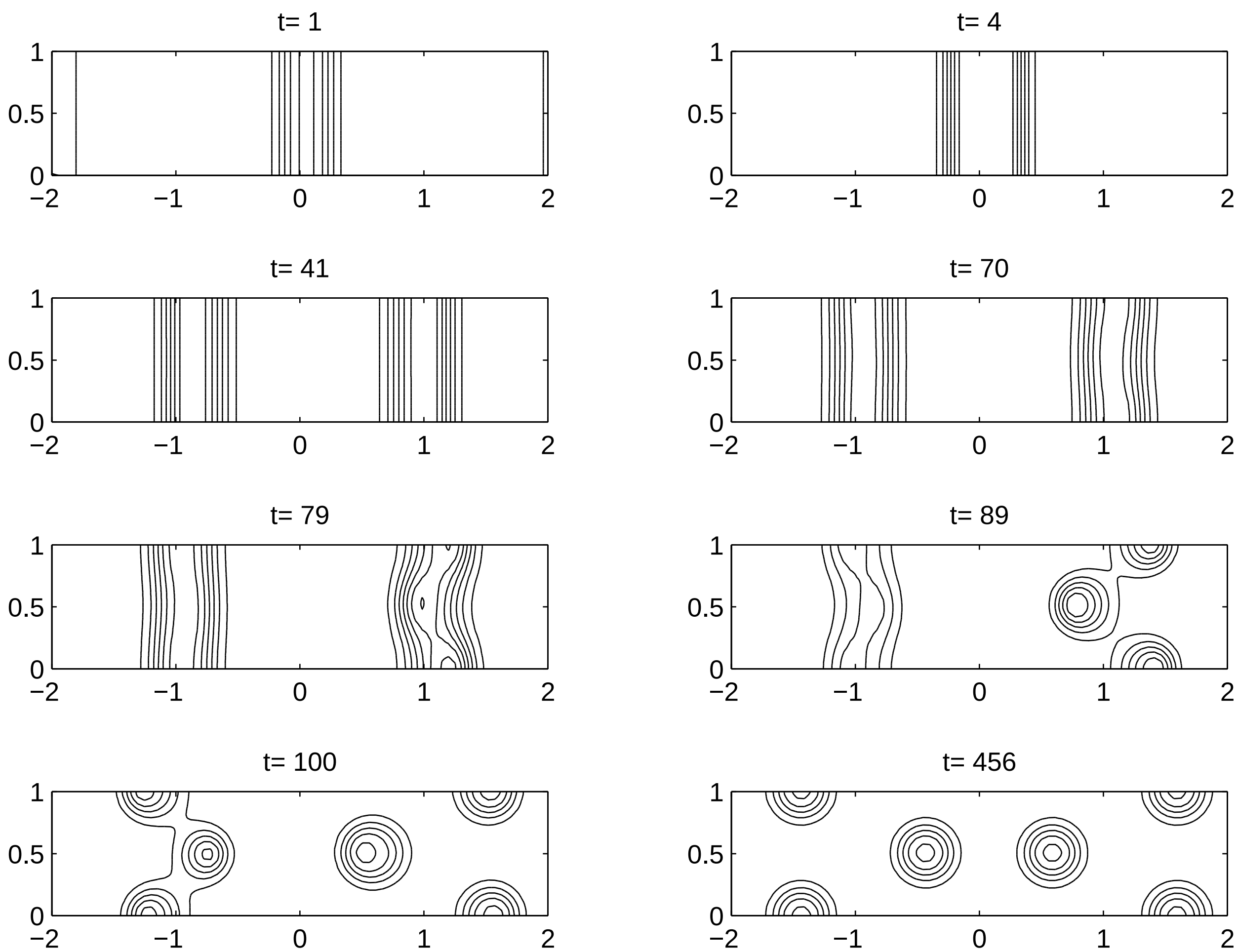
$$v_t = \varepsilon^2 \triangle v - v + Av^2 u$$
$$\tau u_t = D \triangle u - u + 1 - v^2 u$$

- We assume $D \gg \varepsilon^2$

- In 1D solutions are spikes.

- Stability in 1D depends on small $O(\varepsilon^2)$ and large $O(1)$ eigenvalues.

- In 2D: large eigenvalues $\leftrightarrow$ breakup, small eigenvalues $\leftrightarrow$ zigzag istability


t= 1
t= 4
t= 41
t= 70
t= 79
t= 89
t= 100
t= 456
1
0.5
0
−2
−1
0
1
2

Main result. Assume that domain width is of O(1) and

$$\varepsilon^2 \ll D.$$

Then a breakup instability is always present.

Suppose that

$$\varepsilon\sqrt{D} \geq \frac{3}{2z_0}A^2$$

where $z_0 \sim 1.1997$ is a root of

$$z_0 \tanh z_0 = 1,$$

and

$$A \ll O(1).$$

Then and only then there are no zigzag instabilities.

Mesa patterns in GS

When $D = O(\varepsilon^2)$, mesa patterns are possible.

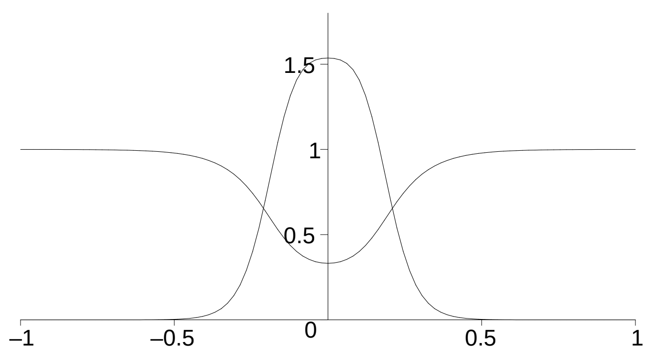
In the case

$$D = \varepsilon^2 \quad \text{and} \quad A = \frac{3}{\sqrt{2}} = 2.1213$$

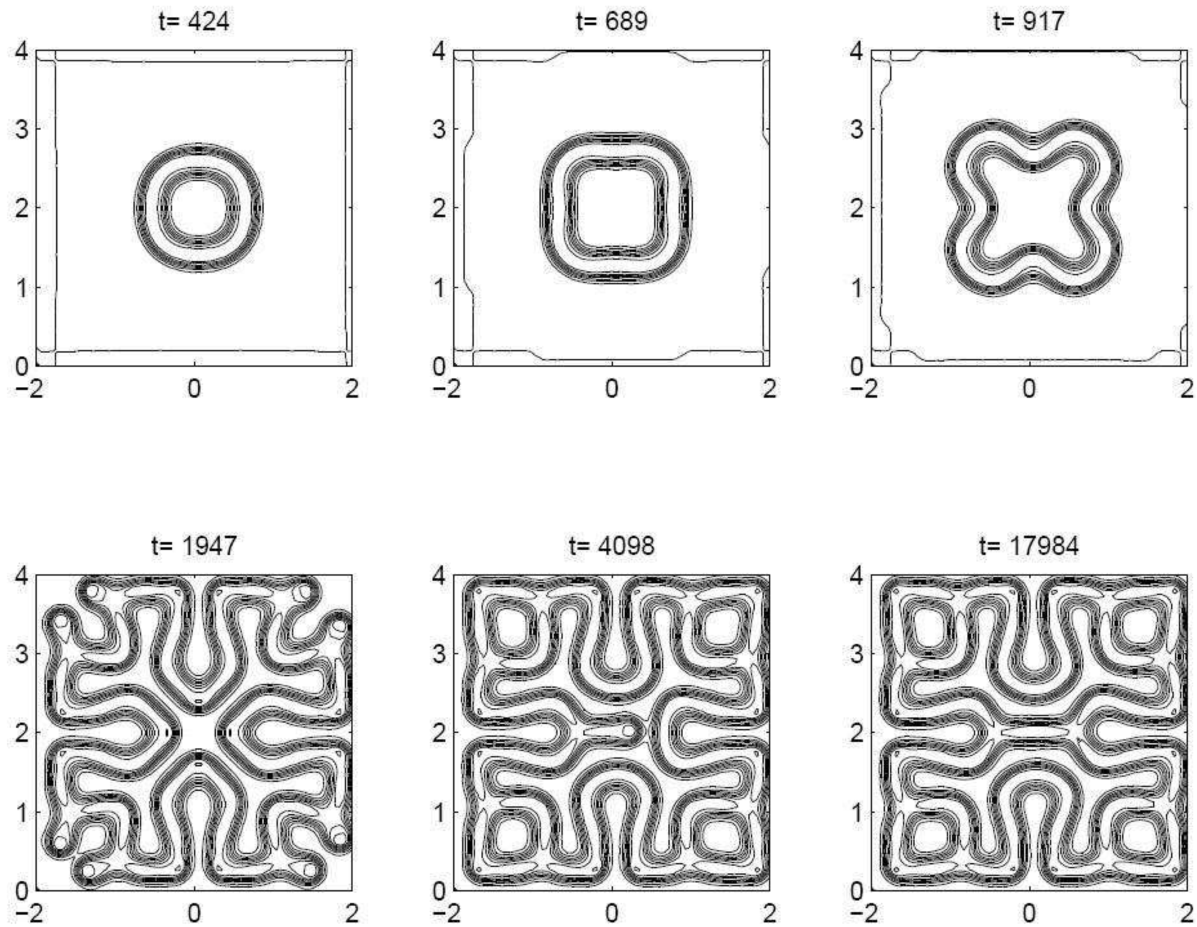
an exact heteroclinic solution exists [HPT, 2000].

When $D - \varepsilon^2 = O(1) \neq 0$, no exact solution is known. However mesa-like patterns are observed numerically:

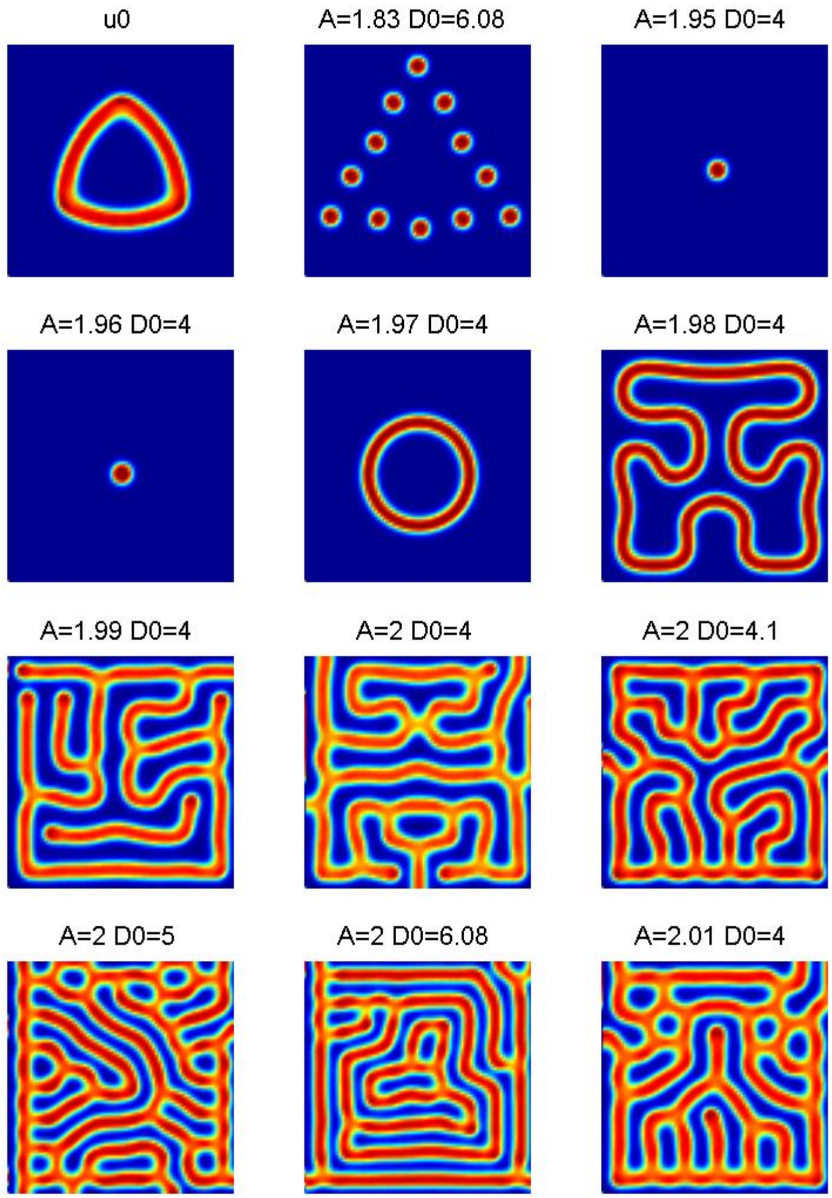
$$A = 2, D = 0.01, \varepsilon = 0.05$$



Numerically, we show that such solution is stable w.r.t breakup but unstable w.r.t. zigzag instabilities.


t= 424
t= 689
t= 917
t= 1947
t= 4098
t= 17984
4
3
2
1
0
−2
0
2

This solution is very sensitive to changes in A, less sensitive to changes in D:



Gierer-Meinhardt model with saturation

$$
\begin{aligned}
A_t &= \varepsilon^2 \Delta A - A + \frac{A^2}{1+\delta A^2}\frac{1}{H} \\
\tau H_t &= D\Delta H - H + A^2,
\end{aligned}
$$

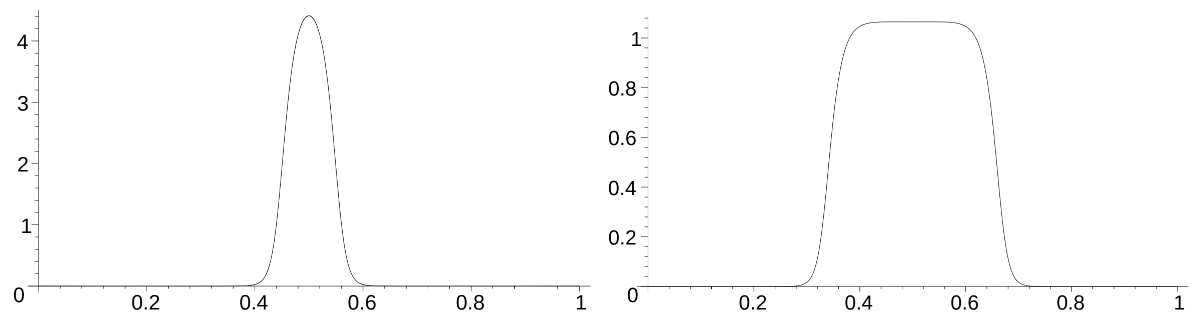
$$\varepsilon \ll 1, \quad D \gg 1.$$

• The limit $\delta \ll 1$ is the usual GM model. Solutions are spikes, always have a breakup instability.

• When $\delta = O(1)$, the solutions are mesas. The length of the mesa and its height are given by

$$l = 0.2003\sqrt{\delta}, \qquad A_{\text{head}} \sim 1.517\frac{1}{\sqrt{\delta}}.$$

Example of mesa



Here, $\varepsilon = 0.01, D = 10$. For left figure, saturation $\delta = 0.1$; for right figure, $\delta = 2$.

Remark: Mesas occur in many other models, such as FitzHugh-Nagumo model (Goldstein, Muraki, Petrich, 1996), Diblock Copolymers (Choksi, Ren, Wei), and the Brusselator.

Eigenvalues are given by:

$$\lambda_{\text{zig}} \sim -m^2\varepsilon^2 + 3.622\frac{\varepsilon}{Dl}\left(\frac{l(1-l)}{2} - \sigma_-\right),$$

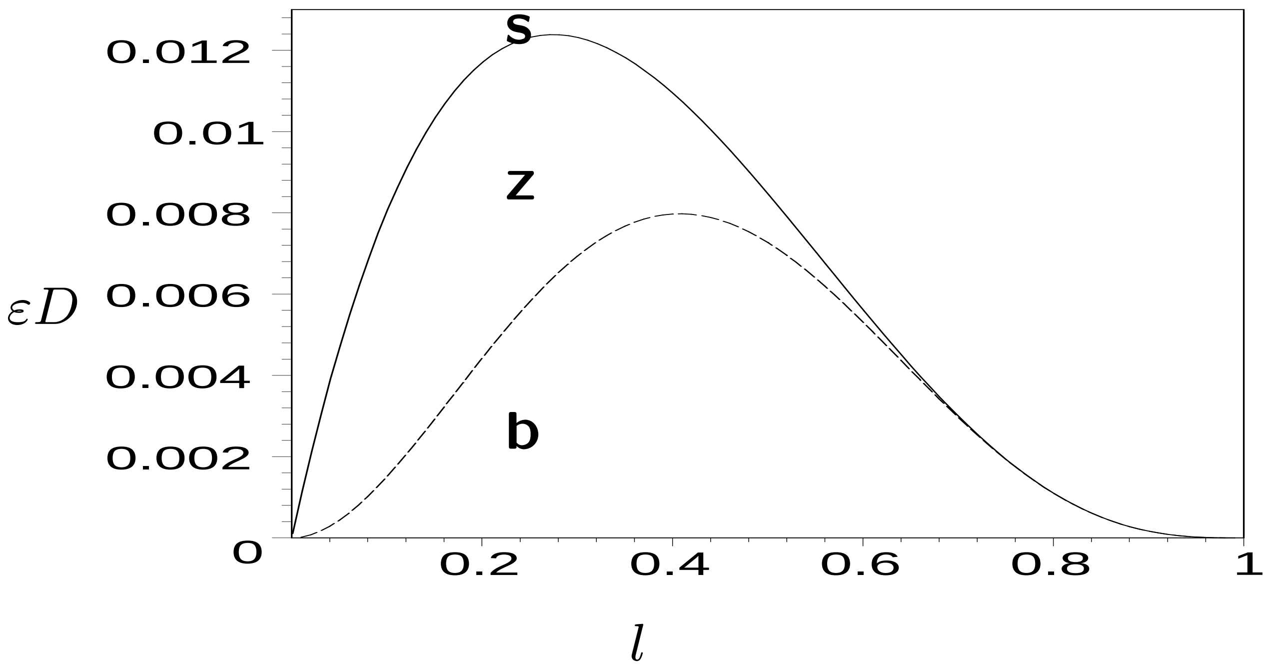
$$\lambda_{\text{break}} \sim -m^2\varepsilon^2 + 3.622\frac{\varepsilon}{Dl}\left(\frac{(1-l)\,l}{2} - \frac{\sigma_+}{1+5.09\frac{\xi}{lD}}\right)$$

where

$$\sigma_+ = \frac{\cosh\frac{\mu(1-l)}{2}}{\mu}\frac{\cosh\frac{\mu l}{2}}{\sinh\frac{\mu}{2}}, \quad \mu = \sqrt{m^2 + \frac{1}{D}}$$

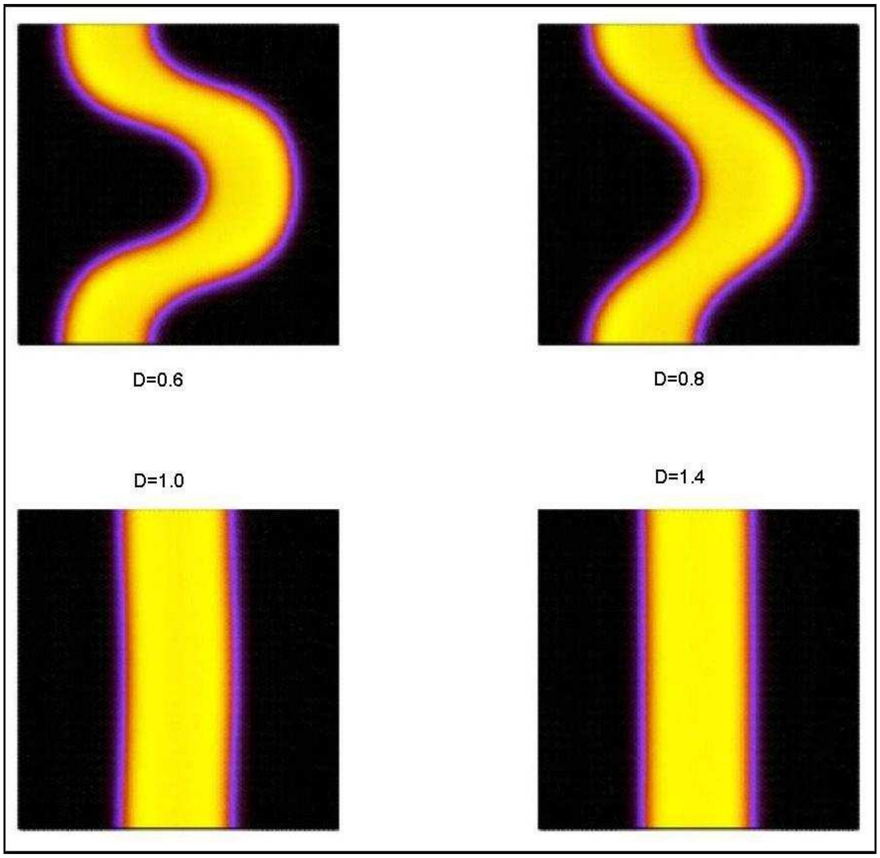
$$\sigma_- = \frac{\cosh\frac{\mu(1-l)}{2}}{\mu}\frac{\sinh\frac{\mu l}{2}}{\cosh\frac{\mu}{2}},$$

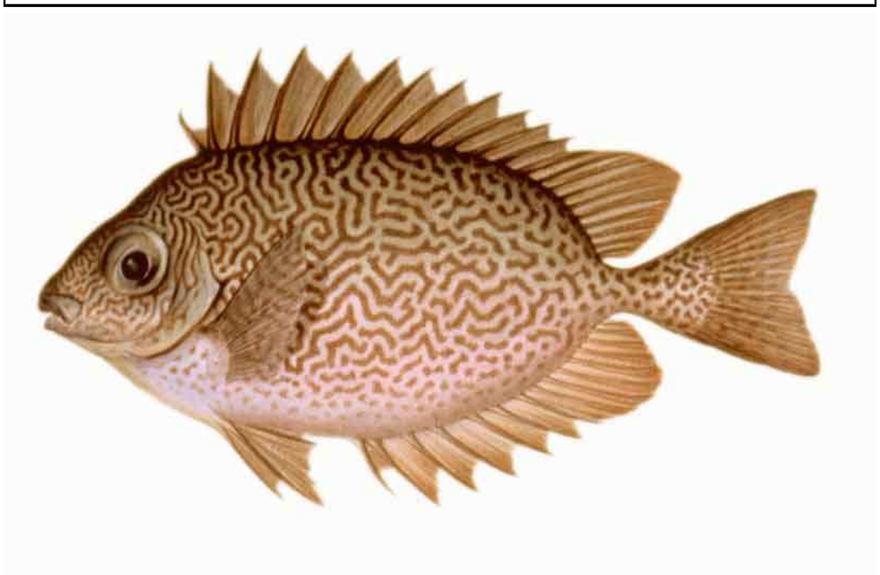
$$\xi = \frac{\sinh\left(\mu\frac{l}{2}\right)}{\mu^2\sinh\left(\frac{\mu}{2}\right)}\cosh\left(\frac{\mu}{2}(l-1)\right)$$



The graph of l versus the maximum value of εD for which an instability can occur. The solid and dotted curves correspond to zigzag and breakup instabilities, respectively.

For example take $l = 0.25, \varepsilon = 0.01$. We get stability if $D = 1.2$, zigzag innstability if $D = 0.8$.


D=0.6
D=0.8
D=1.0
D=1.4



GM model with $D = O(\varepsilon^2)$

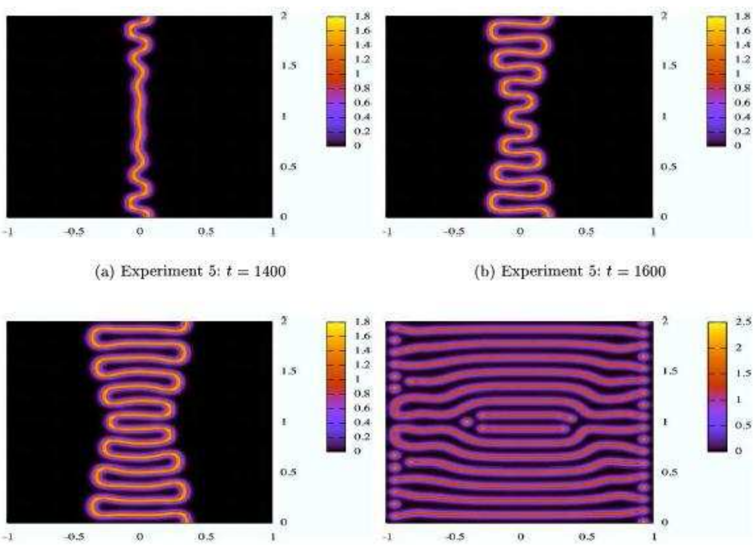
Take $D = D_0 \varepsilon$ and $\delta = 0$. If

$$D_0 < 7.17$$

then a 1-D spike dissapears; leading to pulse splitting. If

$$7.17 < D_0 < 8.06$$

then the stripe is stable w.r.t. breakup instability, but unstable w.r.t. zigzag instability. The final state is a Turing-type pattern.



(a) Experiment 5: $t = 1400$ (b) Experiment 5: $t = 1600$ (c) Experiment 5: $t = 1800$ (d) Experiment 5: $t = 4300$

The Brusselator model

Rate equations:

$$A \xrightarrow{\varepsilon} X, \quad B{+}X \rightarrow Y{+}D, \quad 2X{+}Y \rightarrow 3X, \quad X \xrightarrow{\varepsilon} E.$$

After rescaling, we get a PDE system:

$$\begin{aligned} v_t &= \varepsilon D v_{xx} + Bu - u^2 v, \\ \tau u_t &= \varepsilon D u_{xx} + \varepsilon A + u^2 v - (B + \varepsilon)\, u \end{aligned}$$

Steady state

$$0 = \varepsilon D v_{xx} + Bu - u^2 v,$$
$$0 = \varepsilon D u_{xx} + \varepsilon A + u^2 v - (B + \varepsilon)\, u$$

Let $w = v + u$; then

$$0 = \delta^2 v_{xx} + B\,(w - v) - (w - v)^2\, v,$$
$$0 = D w_{xx} - w + v + A$$

where $\delta^2 = \varepsilon D \ll 0$ and $D \gg 1$. Therefore

$$w \sim w_0$$

is constant to first order; and $\delta^2 v_{xx} = \text{Cubic}(v)$. The **Maxwell line** condition then implies:

$$B = \frac{2}{9} w_0^2.$$

Away from interfaces, $v \sim w_0$ or $v \sim w_0/3$. Near the interface x_0,

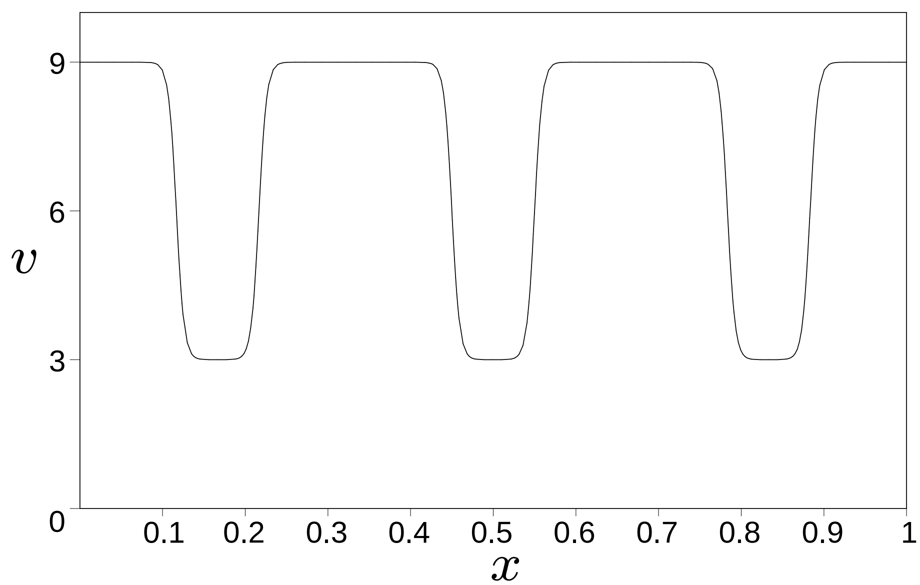
$$v \sim w_0 \frac{2}{3} \pm w_0 \frac{1}{3} \tanh \left(\frac{w_0}{3} \frac{(x - x_0)}{\sqrt{2\varepsilon D}} \right)$$

Suppose $v \sim w_0/3$ on $[0, l]$ and $v \sim w_0$ on $[l, 1]$. Using solvability condition we obtain,

$$w_0 - A = \int_0^1 v = l w_0/3 + (1 - l) w_0$$

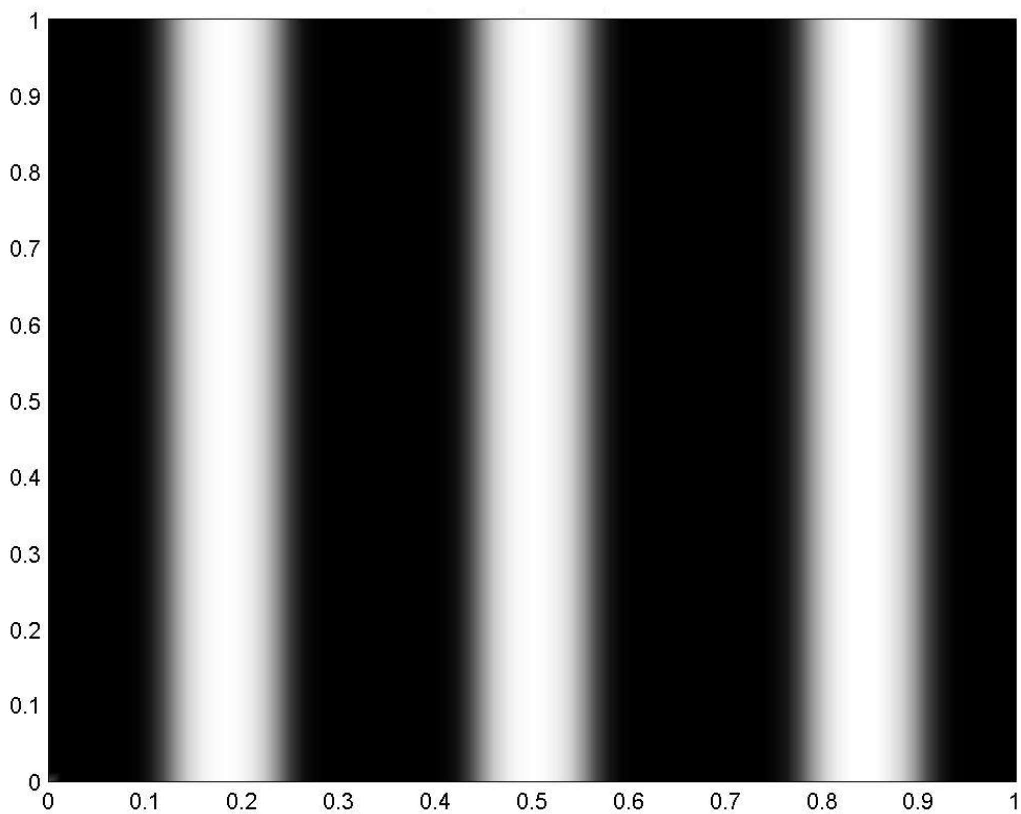
and so

$$l = \frac{A}{\sqrt{2B}}.$$

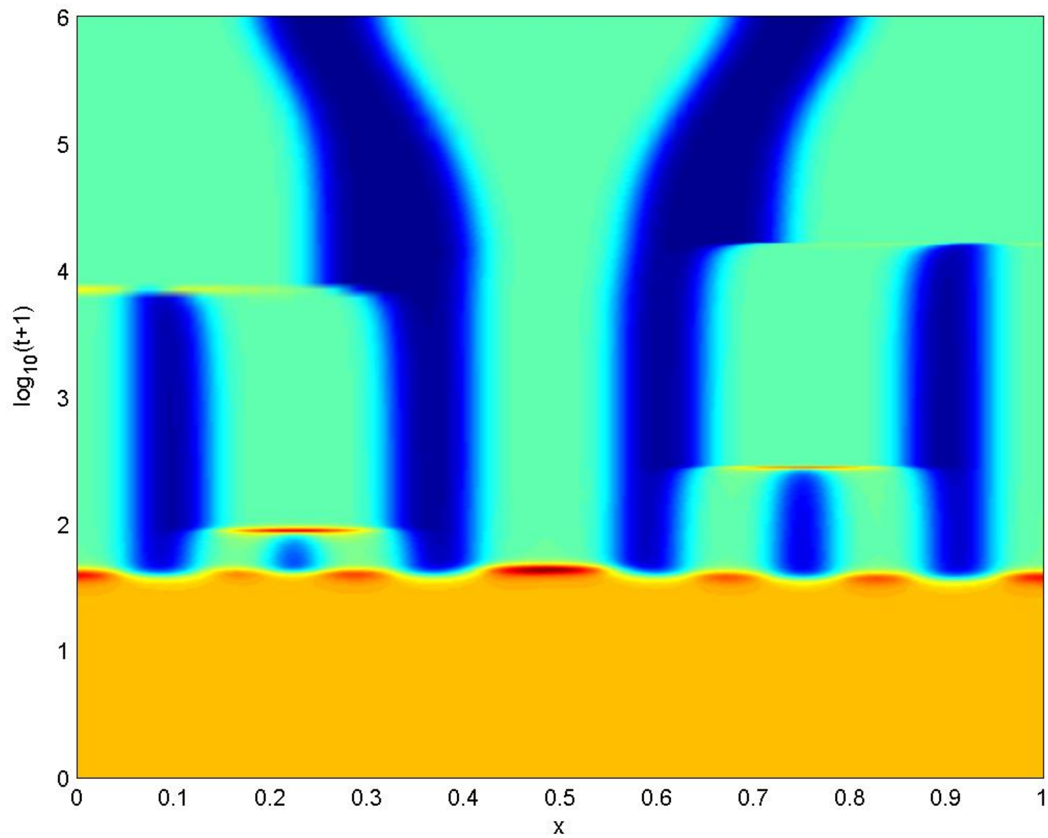


An example of a three-mesa equilibrium state for v. Here, $K = 3$, $A = 2$, $B = 18$, $\varepsilon D = 0.02^2$.

In 2D these mesas can be stable. Analysis is similar to GMS.



Coarsening in 1-D



$A = 1, B = 8,\ \varepsilon = 10^{-4},\ D = 10,\ \tau = 10.$

Conclusions

- Stripes formed from spikes break up
- Stripes formed from mesas can be stable
- Turing patterns can form stripes
- Two different mechanisms to get stripes in GMS model
- Space-filling curves occur in presence of zigzag and absence of breakup