# ON LARGE RING SOLUTIONS FOR GIERER-MEINHARDT SYSTEM IN $\mathbb{R}^3$

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Abstract. We consider the stationary radial solution in the classical Gierer-Meinhardt system in $\mathbb{R}^3$:

$$\begin{cases} \epsilon^2\Delta u - u + \frac{u^2}{v} = 0 & \text{in } \mathbb{R}^3, \\ \Delta v - v + u^2 = 0 & \text{in } \mathbb{R}^3, \\ u, v > 0, u(x), v(x) \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

We prove the existence of a large ring-like solution. This complements an earlier work of Ni and Wei [33] in which the existence of $O(1)$ ring solutions was proved in $\mathbb{R}^2$.

## 1. Introduction

Of concern is the stationary Gierer-Meinhardt system in $\mathbb{R}^3$:

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^2}{v} = 0 & \text{in } \mathbb{R}^3, \\ \Delta v - v + u^2 = 0 & \text{in } \mathbb{R}^3, \\ u > 0, v > 0, u(x), v(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small constant.

Gierer-Meinhardt system was proposed in [14] to model head formation of *hydra*, an animal of a few millimeters in length, made up of approximately 100,000 cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments with *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if the transplanted area is sufficiently far from the (old) head. These observations led to the assumption of the existence of two chemical substances—a *slowly* diffusing activator $u$ and a *rapidly* diffusing inhibitor $v$. The ratio of their diffusion rates, denoted by $\varepsilon$, is assumed to be small.

The Gierer-Meinhardt system falls within the framework of a theory proposed by Turing [35] in 1952 as a mathematical model for the development of complex organisms from a single cell. He speculated that localized peaks in concentration of chemical substances, known as inducers or morphogens, could be responsible for a group of cells developing differently from the surrounding cells. Turing discovered through linear analysis that a large difference in relative size of diffusivities for activating and inhibiting substances carries instability of the homogeneous, constant steady state, thus leading to the presence of nontrivial, possibly stable stationary configurations. Activator-inhibitor systems have been used extensively in the mathematical theory of biological pattern formation [21], [22]. Among them Gierer-Meinhardt system has been the object of extensive mathematical treatment

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in recent years. We refer the reader to two survey articles [25, 42] for a description of progress made and references.

In particular, it has been a matter of high interest to study nonconstant positive steady states, namely, solutions of the following elliptic system

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \Omega, \\ D\Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the exponents $(p, q, m, s)$ satisfy the following condition:

$$p > 1, q > 0, m > 1, s \geq 0, \frac{qm}{(p-1)(1+s)} > 1. \tag{1.3}$$

A first step in solving Problem (1.2) is to study its shadow system, namely, we take $D = +\infty$ first. By suitable scaling, the study of steady-state for the shadow system can be transformed to that of the scalar equation

$$\begin{cases} \varepsilon^2\Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

For problem (1.4), there have been intense works on the construction of a single or multiple spikes. For the case $p$ in subcritical range, we refer the readers to the articles [6], [7], [15], [16], [29], [36], [38] and the references therein, starting with the pioneering works [17], [26], [27], [28], [32]. For the critical exponent case, we refer to the papers [3], [4], [13], [31], and the references therein. A review of the subject up to 2004 can be found in [42].

In the case of finite $D$ and bounded domain case, Takagi [34] first constructed multiple symmetric peaks in the one-dimensional case. In higher dimensional case, Ni and Takagi [29] constructed multiple boundary spikes in the case of axially symmetric domains, assuming that $D$ is large. Multiple interior spikes for finite $D$ case in a bounded two dimensional domain are constructed in [39], [40] and [41]. The stability of multiple spikes as well as the dynamics of spikes are considered in [11], [30], [37], [40], [41] and the references therein.

From now on, we focus on the case of $\Omega = \mathbb{R}^N$. Problem (1.2) has been shown to exhibit single or multiple bump solutions in one or two dimensions. See [8], [9], [11], [12] and the references therein.

A long-standing problem is the existence of radially symmetric bound states in $\mathbb{R}^N$ when $N \geq 3$. (See also page 579 of [42].) Ni and Wei [33] first constructed ring-like solutions for the generalized Gierer-Meinhardt system

$$\begin{cases} \varepsilon^2\Delta u - u + \frac{u^p}{v^q} = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - v + \frac{u^m}{v^s} = 0 & \text{in } \mathbb{R}^N, \\ u > 0, v > 0, u(x), v(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.5}$$

where $(p, q, m, s)$ satisfies, in addition to the usual structural condition (1.3), the following new condition

$$\frac{(N-2)q}{N-1} + 1 < p < q + 1 \text{ if } N \geq 3; 1 < p \leq q + 1 \text{ if } N = 2. \tag{1.6}$$

These are radially symmetric solutions such that $u$ concentrates on an the surface of a ball of a certain $O(1)$ radius as $\varepsilon \to 0$. In the case of $\mathbb{R}^2$, the first two authors showed in [19] that there are solutions with multiple clustered rings.

In $\mathbb{R}^3$, unfortunately the classical Gierer-Meinhardt system (1.1) (i.e. $p = 2, q = 1$), lies at the bordeline case ($p = q + 1$) in the condition (1.6). The purpose of this paper is to give a **confirmative** answer to the existence of radially symmetric bound states of the classical Gierer-Meinhardt system (1.1) in $\mathbb{R}^3$. This seems to be the first radial bound states for Gierer-Meinhardt system. We should remark that as far as non-radial solutions are concerned, in [20], a *smoke-ring* solution was constructed for (1.5) with $N = 3$, provided that $(p, q, m, s)$ satisfy

$$p = q + 1, \quad 1 < m - s < 3. \tag{1.7}$$

This includes the case of the classical GM system (1.1). A smoke-ring solution concentrates on a *circle* in $\mathbb{R}^3$ and is axi-symmetric but not radially symmetric.

Before we state the main result of this paper, we define some notations to be used throughout the paper. Let the Green function be

$$G'' + \frac{2}{r}G' - G + \delta(r - r_0) = 0, \ G'(0, r_0) = 0. \tag{1.8}$$

Its solution is given by

$$G(r, r_0) = \begin{cases} \frac{r_0}{2r}(e^{r-r_0} - e^{-r-r_0}) = \frac{r_0}{r}e^{-r_0}\sinh(r), \ r < r_0, \\ \frac{r_0}{2r}(e^{r_0-r} - e^{-r-r_0}) = \frac{r_0}{r}e^{-r}\sinh(r_0), \ r > r_0. \end{cases} \tag{1.9}$$

Let $w(y)$ be the unique solution for the following ODE:

$$w^{''} - w + w^2 = 0 \text{ in } \mathbb{R}, \ w > 0, \ w(0) = \max_{y\in\mathbb{R}} w(y), w(y) \to 0 \text{ as } |y| \to \infty; \tag{1.10}$$

it is well known that

$$w(y) = \frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right).$$

We now state our main theorem in this paper.

**Theorem 1.1.** *For $\varepsilon$ sufficiently small, problem (1.1) has a solution with the following properties:*

- $u_{\varepsilon,R}, v_{\varepsilon,R}$ *are radially symmetric,*
- $u_{\varepsilon,R} = \frac{1}{6\varepsilon G(r_\varepsilon,r_\varepsilon)}w\left(\frac{r-r_\varepsilon}{\varepsilon}\right)(1 + o(1)),$
- $v_{\varepsilon,R} = \frac{1}{6\varepsilon G(r_\varepsilon,r_\varepsilon)^2}G(r, r_\varepsilon)(1 + o(1))$, *where $r_\varepsilon$ stands for the solution of the following equation*

$$(2r + 1)e^{-2r} = \frac{103}{70}\varepsilon. \tag{1.11}$$

Asymptotically the radius $r_\epsilon$ behaves like

$$r_\epsilon = \frac{1}{2}\log\frac{1}{\epsilon} + O(\log\log\frac{1}{\epsilon}); \tag{1.12}$$

so in fact $r_\varepsilon \to \infty$ as $\varepsilon \to 0$. In contrast, for the general GM model (1.5) and under conditions (1.6), the ring radius was derived in [33]; it was found that when $N = 3$, $r_\varepsilon \to r_0$ where $r_0$ satisfies

$$\frac{p-1}{q} = \frac{e^{2r_0} - 1 - r_0}{e^{2r_0} - 1}$$

It is clear that $r_0 \to +\infty$ as $p \to q + 1$ from the the right; this is indeed consistent with Theorem 1.1.

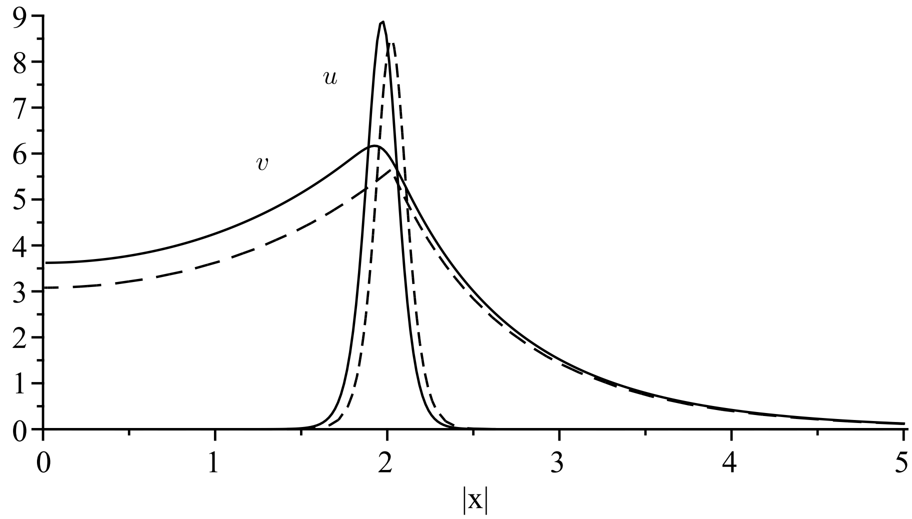


FIGURE 1. The ring solution to (1.1) with $\varepsilon = 0.06$. Dashed curves correspond to the asymptotic profile given in Theorem 1.1

In addition to the rigorous proof below, we also verify the formula (1.11) directly by comparing it to the full numerical solution of (1.1). This is shown in Figure 1 with $\varepsilon = 0.06$. The following table gives the data for a few additional values of $\varepsilon$ :

| $\varepsilon$ | $r_\varepsilon$ (from full numerics) | $r_\varepsilon$ from (1.11 ) | relative error |
|---|---|---|---|
| 0.12 | 1.5414 | 1.5797 | 2.4% |
| 0.06 | 1.9701 | 2.0228 | 2.6% |
| 0.03 | 2.3917 | 2.4472 | 2.2% |

Excellent agreement is observed.

For concreteness, we limit this paper to the classical GM model $p = 2, q = 1, m = 2, s = 0$. However we expect the techniques to apply for more general $p, q, m, s$, provided that $p = q + 1$.

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## 2. OUTLINE OF THE PROOF OF THEOREM 1.1

Our strategy of the proof of the main results is based on the idea of solving the second equation in (1.1) for $v$ and then working with a nonlocal elliptic PDE rather than directly with the system. This procedure is standard that have been used in [9], [19] and [33]. However, for the problem (1.1), the key function $M(t)$ introduced in [33] can be explicitly computed $M(t) = \frac{2}{e^{2t}-1}$. It is easy to see that the function $M(t)$ never possesses a zero point in $t \in (0, +\infty)$, as a result, we can not get the reduced problem solved. Instead, we need to give a more precise formula of the approximate solution.

It's convenient to rescale the problem by replacing the $u$ by $(\varepsilon\xi)^{-1}u$ and $v$ by $(\varepsilon\xi)^{-1}v$, where $\xi > 0$ is to be chosen. (1.1) is transformed into the following radial form:

$$\begin{cases} \varepsilon^2(u'' + \frac{2}{r}u') - u + \frac{u^2}{v} = 0, \\ v'' + \frac{2}{r}v' - v + \frac{u^2}{\varepsilon\xi} = 0, \end{cases} \tag{2.1}$$

where $\xi = \int_0^\infty w^2$ is for convenience and we will give reason for this choice in Appendix A.

We consider a point $t \in (0, +\infty)$ which is a candidate for the location of concentration. Then we come to consider the second equation and by $\mathbb{T}[u^2]$ we denote the unique solution of the equation

$$v'' + \frac{2}{r}v' - v + \frac{u^2}{\varepsilon\xi} = 0. \tag{2.2}$$

By using the Green function defined in (1.9), $\mathbb{T}[u^2]$ can be written in the following way:

$$\mathbb{T}[u^2](r) = \frac{1}{\varepsilon\xi}\int_0^\infty G(r, r')u^2(r')\mathrm{d}r'. \tag{2.3}$$

Once we solved the second equation for $v$ in (2.1) and scaling $r = t + \varepsilon y$, we get the nonlocal PDE for $u$

$$u'' + \frac{2\varepsilon}{t+\varepsilon y}u' - u + \frac{u^2}{\mathbb{T}[u^2](t+\varepsilon y)} = 0, \; y \in (-\frac{t}{\varepsilon}, +\infty), \; u'(-\frac{t}{\varepsilon}) = 0. \tag{2.4}$$

By writing $r' = t + \varepsilon z$, we have

$$v(y) = \frac{1}{\xi}\int_0^\infty G(t+\varepsilon y, t+\varepsilon z)u^2(t+\varepsilon z)\mathrm{d}z. \tag{2.5}$$

We look for a solution to (2.4) in the form $u = w(\frac{r-t}{\varepsilon}) + \varepsilon U_1 + \phi$, where $U_1$ is a function that will be introduced in Appendix A and $\phi$ is the perturbation term. Here we want to say more about the function $U_1$. If we only use $w(\frac{r-t}{\varepsilon}) + \phi$ as our approximate solution, the equation (2.4) would be transformed into a linear equation with respect to $\phi$ of order $O(\varepsilon)$, and the reduced problem of this linear equation is to find the zero point of the function $M(t) = \frac{2}{e^{2t}-1}$. Though $M(t)$ never vanishes in $t \in (0, +\infty)$, thanks to the fast decay property of this function $M(t)$, we still can solve the equation (2.4) by giving a better approximation. Following this idea, we use $U_1$ to cancel out the first order term appeared in the transformed equation of (2.4), in other words, if we use $w(\frac{r-t}{\varepsilon}) + \varepsilon U_1 + \phi$ as our approximate solution, equation (2.4) will be transformed into a linear equation with respect to $\phi$ of order $O(\varepsilon^2)$. Then, by using the Lyapunov-Schmidt reduction, we can reduce this linear equation to a finite dimension problem which can be solved. Formally, we have

$$\mathbb{T}[u^2] = \mathbb{T}[w^2] + 2\varepsilon\mathbb{T}[wU_1] + 2\mathbb{T}[w\phi] + \varepsilon^2\mathbb{T}[U_1^2] + h.o.t., \tag{2.6}$$

where $h.o.t.$ corresponds to the higher order terms.

Using (2.5) and (2.6), we will prove in Section 4 that $v$ can be written

$$v = 1 + \frac{\int_{-\infty}^{\infty} w\phi}{\int_0^{\infty} w^2} + \varepsilon V_1 + h.o.t., \tag{2.7}$$

where $V_1$ represents the $\varepsilon-$order term in the expansion of $v$, we will give a explicit formula for $V_1$ in Appendix A. Then

$$\frac{u^2}{v} = w^2 + 2w\phi - \frac{\int_{-\infty}^{\infty} w\phi}{\int_0^{\infty} w^2} w^2 - \varepsilon V_1 w^2 + 2\varepsilon w U_1 + h.o.t.. \tag{2.8}$$

Substituting all this in (2.4) we obtain the equation for $\phi$

$$\phi'' + \frac{2\varepsilon}{t+\varepsilon y}\phi' - \phi + 2w\phi - \frac{2\int_{-\infty}^{\infty} w\phi}{\int_{-\infty}^{\infty} w^2} w^2 = \mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi], \tag{2.9}$$

where $\mathbb{E}_\varepsilon$ and $\mathbb{M}_\varepsilon[\phi]$ will be explicitly given in Section 5.

Thus we have reduced the problem of finding solutions to (2.1) to a problem of solving (2.9) for $\phi$.

Rather than directly solving problem (2.9), we consider first the following auxiliary problem: given any point $t$, find a function $\phi$ such that for certain constants $\beta_\varepsilon(t)$ the following equation is satisfied

$$\mathcal{L}_{\varepsilon,t}\phi = \mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi] + \beta_\varepsilon w w', \ \int_{\mathbb{R}} \phi w w' = 0, \tag{2.10}$$

where

$$\mathcal{L}_{\varepsilon,t}[\phi] := \phi'' + \frac{2\varepsilon}{t+\varepsilon y}\phi' - \phi + 2w\phi - \frac{2\int_{-\infty}^{\infty} w\phi}{\int_{-\infty}^{\infty} w^2} w^2.$$

We will prove in Section 3 and Section 5 that this problem is uniquely solvable within a class of small functions $\phi$. We will then get a solution of the original problem when the point $t$ is adjusted in such a way that $\beta_\varepsilon(t) = 0$. We show the existence of such a point in Section 6, thereby proved Theorem 1.1. In Section 7, we will give a detail for computing $U_1$ and $V_1$. In Section 8, we list some numerical results that would be used in Section 6 and Section 7.

## 3. LINEAR PROBLEM

This section is devoted to a study of a linear problem, which reduces our problem to a one dimensional problem.

Let $r_1, r_2$ be a positive given number such that

$$r_1 < t < r_2, \ r_1 = 1, \ r_2 = \frac{1}{\varepsilon^2}.$$

We set

$$\mathbb{I}_{\varepsilon,t} := (-\frac{t}{\varepsilon}, \frac{R-t}{\varepsilon}), \tag{3.1}$$

where $R = 3r_2$. Choose two fixed numbers $R_1 = \frac{r_1}{2}$, $R_2 = 2r_2$. Let $\chi(s)$ be a function such that $\chi(s) = 1$ for $s \in [R_1, \frac{R_2+r_2}{2}]$ and $\chi(s) = 0$ for $s < \frac{R_1}{2}$ or $s > R_2$. Set

$$w_{\varepsilon,t}(y) = w(y)\chi(t+\varepsilon y), \quad \tilde{U}_1(y) = U_1(y)\chi(t+\varepsilon y). \tag{3.2}$$

For $u,v\in H_c^1(\mathbb{R}^3)$, we equip them with the following scalar product:

$$(u,v)_\varepsilon=\int_{\mathbb{I}_{\varepsilon,t}}(u'v'+uv)(t+\varepsilon y)^2\mathrm{d}y \tag{3.3}$$

(which is equivalent to the inner product of $H^1(\mathbb{R}^3)$.)

Then orthogonality to the function $w'_{\varepsilon,t}$ with respect to this scalar product is equivalent to the orthogonality to the function

$$Z_{\varepsilon,t}=w'''_{\varepsilon,t}+\frac{2\varepsilon}{t+\varepsilon y}w''_{\varepsilon,t}-w'_{\varepsilon,t} \tag{3.4}$$

in $L^2(\mathbb{I}_{\varepsilon,t})$, equipped with the following scalar product

$$\langle u,v\rangle_\varepsilon=\int_{\mathbb{I}_{\varepsilon,t}}(uv)(t+\varepsilon y)^2\mathrm{d}y. \tag{3.5}$$

(which is equivalent to the inner product of $L^2(\mathbb{R}^3)$.)

Then we consider the following problem: for $h\in L^2\cap L^\infty(\mathbb{I}_{\varepsilon,t})$ being given, find a function $\phi$ satisfying

$$\begin{cases}\mathcal{L}_{\varepsilon,t}[\phi]:=\phi''+\frac{2\varepsilon}{t+\varepsilon y}\phi'-\phi+2w_{\varepsilon,t}\phi-2\frac{\int_{\mathbb{I}_{\varepsilon,t}}w_{\varepsilon,t}\phi}{\int_{\mathbb{I}_{\varepsilon,t}}w^2_{\varepsilon,t}}w^2_{\varepsilon,t}=h+cZ_{\varepsilon,t},\\ \phi'(-\frac{t}{\varepsilon})=0,\quad \langle\phi,Z_{\varepsilon,t}\rangle_\varepsilon=0\end{cases} \tag{3.6}$$

for some constant c.

Before we solve system (3.6), we need the following result which is just the Lemma 5.1 in [33]. For convenience of the readers, we repeat the proof here.

**Lemma 3.1.** *Let $\phi\in C^2(\bar{\mathbb{I}}_{\varepsilon,t})$ satisfy*

$$|\phi''(y)+\frac{2\varepsilon}{t+\varepsilon y}\phi'(y)-\phi(y)|\le c_0e^{-\mu|y|},\quad \phi'(-\frac{t}{\varepsilon})=0,$$

*for some $c_0>0$ and $\mu\in(0,1)$. Then, provided that $\mu>0$ is sufficiently small,*

$$|\phi(y)|\le 2e^2(|\phi(0)|+c_0)e^{-\mu|y|},\quad \forall y\in\mathbb{I}_{\varepsilon,t}.$$

*Proof.* We use a comparison principle. Take $\eta(t)$ a smooth cut-off function such that

$$\eta(t)=1 \text{ for } |t|\le 1,\ \eta(t)=0 \text{ for } |t|\ge 2,\ 0\le\eta\le 1.$$

Now consider the following auxiliary function:

$$\Phi(y)=A[e^{\mu y}+(e^{\mu y_0}-e^{\mu y})\eta(\mu(y+\frac{t}{\varepsilon}))],$$

where

$$y_0=-\frac{t}{\varepsilon}+\frac{1}{\mu},\quad A=2e(|\phi(0)|+c_0).$$

If $y\in(-\frac{t}{\varepsilon},y_0)$, $\Phi(y)=Ae^{\mu y_0}$ and hence

$$\Phi''+\frac{2\varepsilon}{t+\varepsilon y}\Phi'-\Phi=-Ae^{\mu y_0}\le -c_0e^{\mu y}.$$

If $y\in(-\frac{t}{\varepsilon}+\frac{2}{\mu},0)$, $\Phi(y)=Ae^{\mu y}$ and $\frac{\varepsilon}{t+\varepsilon y}\le\frac{\mu}{2}$, hence

$$\Phi''+\frac{2\varepsilon}{t+\varepsilon y}\Phi'-\Phi\le A[2\mu^2-1]e^{\mu y}\le -c_0e^{\mu y}$$

provided that $\mu$ is sufficiently small. Finally it is easy to see that for $y \in (y_0, y_0 + \frac{1}{\mu})$

$$e^{\mu y_0} \le e^{\mu y} \le e e^{\mu y_0}, \quad \Phi(y) \ge A e^{\mu y_0}, \quad \frac{\mu}{2} \le \frac{\varepsilon}{t + \varepsilon y} \le \mu;$$

hence

$$\begin{aligned} \Phi'' + \frac{2\varepsilon}{t+\varepsilon y}\Phi' - \Phi \le & CA(\mu^2) e^{\mu y} - A e^{\mu y_0} \\ \le & CA(\mu^2) e^{\mu y} - \frac{A}{e} e^{\mu y} \le -c_0 e^{\mu y} \end{aligned}$$

provided that $\mu$ is sufficiently small. Here $C$ is a positive constant.

In any case, we have that for $y \in (-\frac{t}{\varepsilon}, 0)$, $\Phi(y)$ satisfies

$$\Phi^{''} + \frac{2\varepsilon}{t+\varepsilon y}\Phi^{'} - \Phi \le -c_0 e^{\mu y}, \quad \Phi^{'}(-\frac{t}{\varepsilon}) = 0, \quad \Phi(0) \ge |\phi(0)|. \tag{3.7}$$

Combining (3.7) with the hypothesis we obtain

$$(\Phi - \phi)''(y) + \frac{2\varepsilon}{t+\varepsilon y}(\Phi - \phi)'(y) - (\Phi - \phi)(y) \le 0, \quad \forall y \in [-\frac{t}{\varepsilon}, 0] \tag{3.8}$$

and

$$(\Phi - \phi)(0) > 0, \quad (\Phi - \phi)'(-\frac{t}{\varepsilon}) = 0,$$

we claim that $(\Phi - \phi)(y) \ge 0$ for $y \in [-\frac{t}{\varepsilon}, 0)$. Assume the contrary, if we call $\bar{y}$ the minimum point of $\Phi - \phi$ in $[-\frac{t}{\varepsilon}, 0)$, then it would be $(\Phi - \phi)(\bar{y}) < 0$ and $(\Phi - \phi)'(\bar{y}) = 0$, $(\Phi - \phi)''(\bar{y}) \ge 0$, in contradiction with (3.8). Hence we have proved that $\phi \le \Phi$ in $[-\frac{t}{\varepsilon}, 0]$. On the other hand by (3.8) and the hypothesis we also get

$$(\Phi + \phi)''(y) + \frac{2\varepsilon}{t+\varepsilon y}(\Phi + \phi)'(y) - (\Phi + \phi)(y) \le 0 \quad \forall y \in [-\frac{t}{\varepsilon}, 0]$$

and $(\Phi + \phi)(0) > 0$, $(\Phi + \phi)'(-\frac{t}{\varepsilon}) = 0$. Proceeding as before we conclude $\phi \ge -\Phi$ in $[-\frac{t}{\varepsilon}, 0]$.

For $y \in [0, \frac{R-t}{\varepsilon})$, we use $\hat{\Phi}(y) = A\frac{\cosh(\mu(\frac{R-t}{\varepsilon} - y))}{\cosh(\frac{\mu(R-t)}{\varepsilon})}$ as comparison function. Note that $\hat{\Phi}(0) = A, \hat{\Phi}'(\frac{R-t}{\varepsilon}) = 0$. It is easy to see that $\frac{A}{2}e^{-\mu y} < \hat{\Phi}(y) < 2Ae^{-\mu y}$ and hence

$$\hat{\Phi}'' + \frac{2\varepsilon}{t+\varepsilon y}\hat{\Phi}' - \hat{\Phi} \le -c_0 e^{-\mu y}$$

provided that $\mu$ is sufficiently small. By repeating the previous argument we obtain $|\phi| \le \hat{\Phi}$ in $[0, \frac{R-t}{\varepsilon})$ and the conclusion follows. □

Let $\mu \in (0, \frac{1}{10})$ be a small number such that Lemma 3.1 holds. For every function $\phi : \mathbb{I}_{\varepsilon,t} \to \mathbb{R}$, we define

$$\|\phi\|_* = \|e^{\mu\langle y\rangle}\phi(y)\|_{\mathbb{I}_{\varepsilon,t}}, \quad \langle y\rangle = (1+y^2)^{\frac{1}{2}}. \tag{3.9}$$

Since $\frac{2\varepsilon}{t+\varepsilon y}\tilde{U}_0'' = O(\varepsilon)e^{-|y|}$, we obtain

$$Z_{\varepsilon,t}(y) = w''' - w' + O(\varepsilon)e^{-\mu\langle y\rangle} = -2ww' + O(\varepsilon)e^{-\mu\langle y\rangle} \tag{3.10}$$

uniformly for $t \in [R_1, R_2]$.

The following proposition provides a priori estimates of $\phi$ satisfying (3.6).

**Proposition 3.2.** *Let $(\phi, c)$ satisfy (3.6). Then for $\varepsilon$ sufficiently small, $r_0 \in [R_1, R_2]$, we have*

$$\|\phi\|_* \leq C\|h\|_*, \tag{3.11}$$

*where $C$ is a positive constant depending on $R, N$ only.*

**Remark**. A more precise inequality should be

$$\|\phi\|_* \leq C\|h^\perp\|_*, \text{ where } h^\perp = h - \frac{\langle h, Z_{\varepsilon,t}\rangle}{\langle Z_{\varepsilon,t}, Z_{\varepsilon,t}\rangle} Z_{\varepsilon,t}. \tag{3.12}$$

*Proof.* We prove the inequality by contradiction. Arguing by contradiction there exists sequence $\varepsilon_k \to 0$, $t_k \in [R_1, R_2]$ and a sequence of functions $\phi_{\varepsilon_k,t_k}$ satisfying (3.6) such that the following holds

$$\|\phi_{\varepsilon_k,t_k}\|_* = 1, \ \|h_k\|_* = o(1), \int_{\mathbb{I}_{\varepsilon_k,t_k}} \phi_{\varepsilon_k,t_k} Z_{\varepsilon_k,t_k}(t_k + \varepsilon_k y)^2 \mathrm{d}y = 0.$$

For simplicity of notation, we drop the subindex $k$.

Multiplying the first equation of (3.6) by $w'_{\varepsilon,t}$ and integrating over $\mathbb{I}_{\varepsilon,t}$, we obtain that

$$c\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t} w'_{\varepsilon,t} = -\int_{\mathbb{I}_{\varepsilon,t}} h w'_{\varepsilon,t} + \int_{\mathbb{I}_{\varepsilon,t}} (\mathcal{L}_{\varepsilon,t}[\phi_{\varepsilon,t}] w'_{\varepsilon,t}). \tag{3.13}$$

The left-hand side of (3.13) equals $c(-\int_{\mathbb{R}} p w^{p-1} w'^2 + o(1))$ because of (3.10). The first term on the right-hand side of (3.13) can be estimated by

$$\int_{\mathbb{I}_{\varepsilon,t}} h w'_{\varepsilon,t} = O(\|h\|_*),$$

where we have used the fact that $w$ is exponentially decay.

The last term equals

$$\begin{aligned}
\int_{\mathbb{I}_{\varepsilon,t}} (\mathcal{L}_{\varepsilon,t}[\phi_{\varepsilon,t}]) w'_{\varepsilon,t} =& \int_{\mathbb{I}_{\varepsilon,t}} \left[\phi''_{\varepsilon,t} + \frac{2\varepsilon}{t + \varepsilon y}\phi'_{\varepsilon,t} - \phi_{\varepsilon,t} + 2w_{\varepsilon,t}\phi_{\varepsilon,t}\right] w'_{\varepsilon,t} \\
&- 2\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}\phi_{\varepsilon,t}}{\int_{\mathbb{I}_{\varepsilon,t}} w^2_{\varepsilon,t}} \int_{\mathbb{I}_{\varepsilon,t}} w^2_{\varepsilon,t} w'_{\varepsilon,t} \\
=& \int_{\mathbb{I}_{\varepsilon,t}} \left[w'''_{\varepsilon,t} - w'_{\varepsilon,t} + 2w_{\varepsilon,t} w'_{\varepsilon,t}\right] \phi_{\varepsilon,t} + O(\varepsilon\|\phi_{\varepsilon,t}\|_*) \\
=& o(\|\phi_{\varepsilon,t}\|_*).
\end{aligned}$$

Hence we obtain that

$$|c| = O(\|h\|_*) + o(\|\phi_{\varepsilon,t}\|_*), \quad \|h + cZ_{\varepsilon,t}\|_* = o(1). \tag{3.14}$$

Next, we claim that $\|\phi_{\varepsilon,t}(y)\| \to 0$ in any compact interval of $\mathbb{R}$. In fact, we consider $\overline{\phi}_\varepsilon(y) = \phi_{\varepsilon,t}\chi(t + \varepsilon y)$ where $\chi$ is the cut-off function introduced at the beginning of the this section. Then since $\|\phi_{\varepsilon,t}\|_* = 1$, it is easy to see that $\|\overline{\phi}_\varepsilon\|_{H^2} \leq C$ and hence $\overline{\phi}_\varepsilon \to \phi_0$ weakly in $H^2(\mathbb{R})$ and $\phi_0$ satisfies

$$\phi''_0 - \phi_0 + 2w\phi_0 - \frac{2\int_{\mathbb{R}} w\phi_0}{\int_{\mathbb{R}} w^2} w^2 = 0, \ |\phi_0| \leq Ce^{-\mu|y|}.$$

By Lemma 4.2 in [33], we must have $\phi_0 = cw'$. On the other hand, $\int_{\mathbb{I}_{\varepsilon,t}} \phi_{\varepsilon,t} Z_{\varepsilon,t}(t+\varepsilon y)^2 \mathrm{d}y = 0$ and hence $\int_{\mathbb{R}} \phi_0 w w' = 0$ which implies that $c = 0$ and hence $\phi_{\varepsilon,t} \to 0$ in any compact interval of $\mathbb{R}$. This shows that

$$\|w_{\varepsilon,t}\phi_{\varepsilon,t}\|_* = \sup_{y\in\mathbb{I}_{\varepsilon,t}} |e^{\mu\langle y\rangle} w_{\varepsilon,t}(y)\phi_{\varepsilon,t}(y)| = o(1). \tag{3.15}$$

On the other hand, by Lebesgue's Dominated Convergence Theorem, we have that

$$\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}\phi_{\varepsilon,t} \to 0$$

which implies

$$\|\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}\phi_{\varepsilon,t}}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^2} w_{\varepsilon,t}\|_* = o(1). \tag{3.16}$$

Thus we have arrived at the following situation $\phi_{\varepsilon,t}$ satisfies

$$\phi_{\varepsilon,t}'' + \frac{2\varepsilon}{t+\varepsilon y}\phi_{\varepsilon,t}' - \phi_{\varepsilon,t} = o(e^{\mu\langle y\rangle}),\ \phi_{\varepsilon,t}'(-\frac{t}{\varepsilon}) = 0. \tag{3.17}$$

Since $\phi_{\varepsilon,t} \to 0$ in any compact interval, $\phi_{\varepsilon,t}(0) = o(1)$. Applying Lemma 3.1, we conclude that $\phi_{\varepsilon,t} = o(e^{\mu|y|})$. A contradiction to the assumption that $\|\phi\|_* = 1$. This proves the proposition. □

Finally, we have

**Proposition 3.3.** *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0, t \in [R_1, R_2]$, given any $h \in L^2(\mathbb{I}_{\varepsilon,t}) \cap L^\infty(\mathbb{I}_{\varepsilon,t})$, there exists a unique pair $(\phi, c)$ such that the following hold:*

$$\mathcal{L}_{\varepsilon,t}[\phi] = h + cZ_{\varepsilon,t},\ \phi'(-\frac{t}{\varepsilon}) = 0,\ \langle \phi, Z_{\varepsilon,t}\rangle_\varepsilon = 0. \tag{3.18}$$

*Moreover, we have*

$$\|\phi\|_* \le C\|h\|_*. \tag{3.19}$$

*Proof.* The existence follows from Fredholm alternatives. To this end, let us set

$$\mathcal{H} = \{u \in H^1(B_R) \mid (u, w_{\varepsilon,t}')_\varepsilon = 0\}.$$

Observe that $\phi$ solves (3.18) if only if $\phi \in H^1(B_R)$ satisfies

$$\int_{B_R} (\nabla\phi\nabla\psi + \phi\psi) - 2\langle w_{\varepsilon,t}\phi, \psi\rangle_\varepsilon - 2\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^2}\langle w_{\varepsilon,t}^2, \psi\rangle_\varepsilon = \langle h, \psi\rangle_\varepsilon,\ \forall \psi \in H^1(B_R).$$

This equation can be rewritten in the following form:

$$\phi + \mathcal{S}(\phi) = \overline{h}, \tag{3.20}$$

where $\mathcal{S}$ is a linear compact operator from $\mathcal{H}$ to $\mathcal{H}$, $\overline{h} = (\Delta - 1)^{-1}(h^\perp) \in \mathcal{H}$ and $\phi \in \mathcal{H}$.

Using Fredholm's alternatives, we will show Eq.(3.20) has a unique solvable solution for each $\overline{h}$ by proving that the equation has a unique solution for $\overline{h} = 0$, i.e., $h^\perp = 0$. To this end, we assume the contrary. That is, there exists $(\phi, c)$ such that

$$\mathcal{L}_{\varepsilon,t}[\phi] = cZ_{\varepsilon,t},\ \phi'(-\frac{t}{\varepsilon}) = 0,\ \langle \phi, Z_{\varepsilon,t}\rangle_\varepsilon = 0. \tag{3.21}$$

From (3.21), it is easy to see that $\|\phi\|_* < +\infty$. So without loss of generality, we may assume $\|\phi\|_* = 1$. But then this contradicts (3.12). □

## 4. Study of the operator $\mathbb{T}[h]$

In this section, we study the operator $\mathbb{T}[h]$, where we choose $h$ to be

$$h = (w_{\varepsilon,t}(\frac{r-t}{\varepsilon}) + \varepsilon\tilde{U}_1(\frac{r-t}{\varepsilon}) + \phi(\frac{r-t}{\varepsilon}))^2,\ \|\phi\|_* = O(\varepsilon^\sigma),\ \sigma > 1. \tag{4.1}$$

According the choice of $h$ and the definition of operator $\mathbb{T}$, we have

$$\mathbb{T}[h](y) = \frac{1}{\xi}\int_0^\infty G(t+\varepsilon y, t+\varepsilon z)(w_{\varepsilon,t}(z) + \varepsilon\tilde{U}_1(z) + \phi(z))^2 \mathrm{d}z,$$

where $\xi = \int_0^\infty w^2$. From Appendix A, we can easily prove that $U_1$ and $w$ are exponential decay, therefore, we conclude

$$\begin{aligned}\mathbb{T}[h](y) =& \frac{1}{\xi}\int_{-\infty}^\infty G(r_0+\varepsilon y, r_0+\varepsilon z)(w_{\varepsilon,t}(z) + \varepsilon\tilde{U}_1(z) + \phi(z))^2 \mathrm{d}z + o(\varepsilon^3)\\ =& \frac{1}{\xi}\int_{-\infty}^\infty G(r_0+\varepsilon y, r_0+\varepsilon z)(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z + o(\varepsilon^3).\end{aligned} \tag{4.2}$$

Define $a$ by

$$e^{-2t} = a\varepsilon. \tag{4.3}$$

Then we write

$$G(t+\varepsilon y, t+\varepsilon z) = G_0 + \begin{cases} \varepsilon G_1^- + \varepsilon^2 G_2^-, & y < z,\\ \varepsilon G_1^+ + \varepsilon^2 G_2^+, & y > z,\end{cases} \tag{4.4}$$

where:

$$\begin{aligned} G_0 &= \frac{1}{2},\ G_1^- = -\frac{a}{2} + (y-z)(\frac{1}{2} - \frac{1}{2t}),\ G_1^+ = -\frac{a}{2} + (y-z)(-\frac{1}{2} - \frac{1}{2t}),\\ G_2^- &= \frac{(y-z)^2}{4t^2}(y(t^2-2t+2) + z(-t^2+2t)) + \frac{a}{2t}(y(t+1) + z(t-1)),\\ G_2^+ &= \frac{(y-z)^2}{4t^2}(y(t^2+2t+2) + z(-t^2-2t)) + \frac{a}{2t}(y(t+1) + z(t-1)).\end{aligned}$$

Substituting (4.3) and (4.4) into (4.2),

$$\begin{aligned}\mathbb{T}[h](y) =& \frac{1}{\xi}\int_{-\infty}^\infty G_0(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z\\ &+ \frac{1}{\xi}\int_{-\infty}^y \varepsilon G_1^+(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z\\ &+ \frac{1}{\xi}\int_y^\infty \varepsilon G_1^-(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z\\ &+ \frac{1}{\xi}\int_{-\infty}^y \varepsilon^2 G_2^+(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z\\ &+ \frac{1}{\xi}\int_y^\infty \varepsilon^2 G_2^-(w(z) + \varepsilon U_1(z) + \phi(z))^2 \mathrm{d}z + o(\varepsilon^3)\\ =& A_1 + A_2 + A_3 + A_4 + A_5 + o(\varepsilon^3).\end{aligned} \tag{4.5}$$

We study the terms $A_i$ $(i=1,2,3,4,5)$ respectively in the following. For $A_1$, we have

$$\begin{aligned}A_1=&\frac{1}{\xi}\int_{-\infty}^{\infty}G_0(w^2+2w\phi+2\varepsilon wU_1+2\varepsilon U_1\phi+\varepsilon^2U_1^2+\phi^2)\\=&1+\frac{1}{\xi}\int_{-\infty}^{\infty}w\phi+\frac{\varepsilon}{\xi}\int_{-\infty}^{\infty}(wU_1+U_1\phi)+\frac{\varepsilon^2}{2\xi}\int_{-\infty}^{\infty}U_1^2\\&+O(\varepsilon^{1+\sigma}+\|\phi\|_*^2),\end{aligned}\tag{4.6}$$

where we have used $G_0=\frac{1}{2}$ and $\|\phi\|_*\le C\varepsilon^{\sigma},\sigma>1$. For $A_2$ and $A_3$, we have

$$\begin{aligned}A_2+A_3=&\frac{\varepsilon}{\xi}(\int_{-\infty}^{y}G_1^+w^2+\int_{y}^{\infty}G_1^-w^2)+\frac{\varepsilon^2}{\xi}(\int_{-\infty}^{y}2G_1^+wU_1+\int_{y}^{\infty}2G_1^-wU_1)\\&+O(\varepsilon^{1+\sigma}+\|\phi\|_*^2).\end{aligned}\tag{4.7}$$

For $A_4$ and $A_5$, we have

$$A_4+A_5=\frac{\varepsilon^2}{\xi}(\int_{-\infty}^{y}G_2^+w^2+\int_{y}^{\infty}G_2^-w^2)+O(\varepsilon^{1+\sigma}+\|\phi\|_*^2).\tag{4.8}$$

Combining (4.6)-(4.8), we get

$$\begin{aligned}\mathbb{T}[h]=&1+\frac{1}{\xi}\int_{-\infty}^{\infty}w\phi+\varepsilon\Theta_1+\frac{\varepsilon^2}{2\xi}\int_{-\infty}^{\infty}U_1^2+\frac{\varepsilon^2}{\xi}\int_{-\infty}^{y}(2G_1^+wU_1+G_2^+w^2)\\&+\frac{\varepsilon^2}{\xi}\int_{y}^{\infty}(2G_1^-wU_1+G_2^-w^2)+O(\varepsilon^{1+\sigma}+\|\phi\|_*^2),\end{aligned}\tag{4.9}$$

where

$$\Theta_1=\frac{1}{\xi}(\int_{-\infty}^{\infty}wU_1+\int_{-\infty}^{y}G_1^+w^2+\int_{y}^{\infty}G_1^-w^2).$$

From Appendix A, we can see $\Theta_1=V_1$. Summarizing the results, we obtain the following lemma:

**Lemma 4.1.** *For $r=t+\varepsilon y$, we have*

$$\begin{aligned}\mathbb{T}[(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2](t+\varepsilon y)=&1+\frac{1}{\xi}\int_{-\infty}^{\infty}w\phi+\varepsilon V_1+\frac{\varepsilon^2}{2\xi}\int_{-\infty}^{\infty}U_1^2\\&+\frac{\varepsilon^2}{\xi}\int_{-\infty}^{y}(2G_1^+wU_1+G_2^+w^2)\\&+\frac{\varepsilon^2}{\xi}\int_{y}^{\infty}(2G_1^-wU_1+G_2^-w^2)\\&+O(\varepsilon^{1+\sigma}+\|\phi\|_*^2).\end{aligned}\tag{4.10}$$

## 5. A nonlinear problem

In this section, we solve the following system of equation for $(\phi, \beta)$:

$$\begin{aligned}(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)'' + \frac{2\varepsilon}{t+\varepsilon y}(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)' - (w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)\\ + \frac{(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)^2}{\mathbb{T}[(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)^2]} = \beta Z_{\varepsilon,t},\end{aligned} \tag{5.1}$$

with the following constrained condition

$$\phi'(-\frac{t}{\varepsilon}) = 0, \quad \int_{\mathbb{I}_{\varepsilon,t}} \phi Z_{\varepsilon,t}(t+\varepsilon y)^2 \mathrm{d}y = 0. \tag{5.2}$$

The main result in this section is to show the following proposition

**Proposition 5.1.** *For $t \in [R_1, R_2]$ and $\varepsilon$ sufficiently small, there exists a unique pair $(\phi_{\varepsilon,t}, \beta_{\varepsilon,}(t))$ satisfying (5.1) and (5.2). Furthermore, $(\phi_{\varepsilon,t}, \beta_{\varepsilon,}(t))$ is continuous in $t$ and we have the following estimate*

$$\|\phi_{\varepsilon,t}\|_* \le \varepsilon^\sigma, \tag{5.3}$$

*where $\sigma \in (1,2)$ is a constant.*

*Proof.* We write (5.1) in the following form

$$\mathcal{L}_{\varepsilon,t}[\phi] = \mathbb{E}_\varepsilon + \mathbb{M}_\varepsilon[\phi] + \beta Z_{\varepsilon,t}, \tag{5.4}$$

where

$$\mathbb{E}_\varepsilon = -\varepsilon(\tilde{U}_1'' - \tilde{U}_1 + 2w_{\varepsilon,t}\tilde{U}_1 + \frac{2}{t}w_{\varepsilon,t}' - V_1 w_{\varepsilon,t}^2) - (w_{\varepsilon,t}'' - w_{\varepsilon,t} + w_{\varepsilon,t}^2), \tag{5.5}$$

and

$$\begin{aligned}\mathbb{M}_\varepsilon[\phi] = &- \Bigg[\frac{(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)^2}{\mathbb{T}[(w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi)^2]} - w_{\varepsilon,t}^2 + 2\frac{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}\phi}{\int_{\mathbb{I}_{\varepsilon,t}} w_{\varepsilon,t}^2} w_{\varepsilon,t}^2 - 2w_{\varepsilon,t}\phi + \varepsilon V_1 w_{\varepsilon,t}^2\\ &- 2\varepsilon w_{\varepsilon,t}\tilde{U}_1\Bigg] + (\frac{2\varepsilon}{t} - \frac{2\varepsilon}{t+\varepsilon y})w_{\varepsilon,t}' - \frac{2\varepsilon^2}{t+\varepsilon y}\tilde{U}_1'\\ = &\ \mathbb{M}_\varepsilon^1[\phi] + \mathbb{M}_\varepsilon^2[\phi] + \mathbb{M}_\varepsilon^3[\phi].\end{aligned} \tag{5.6}$$

For $\mathbb{E}_\varepsilon$, it is easy to see

$$\begin{aligned}\mathbb{E}_\varepsilon = &-\varepsilon(\tilde{U}_1'' - \tilde{U}_1 + 2w_{\varepsilon,t}\tilde{U}_1 + \frac{2}{t}w_{\varepsilon,t}' - V_1 w_{\varepsilon,t}^2) - (w_{\varepsilon,t}'' - w_{\varepsilon,t} + w_{\varepsilon,t}^2)\\ = &-\varepsilon(U_1'' - U_1 + 2wU_1 + \frac{2}{t}w' - V_1 w^2) - (w'' - w + w^2) + O(\varepsilon^3 e^{-\mu\langle y\rangle}),\end{aligned} \tag{5.7}$$

where we have used the fact that $U_1$, $w$ are exponential decay and $0 < \mu < \frac{1}{10}$. From Appendix A, we see $U_1$ is just the solution to the following equation

$$U_1'' - U_1 + 2wU_1 + \frac{2}{t}w' - V_1 w^2 = 0.$$

While $w'' - w + w^2 = 0$ holds for the setting of $w$. Hence, we conclude

$$\|\mathbb{E}_\varepsilon\|_* \le C\varepsilon^3. \tag{5.8}$$

For $\mathbb{M}_\varepsilon$, using (4.10), we can write $\frac{(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2}{\mathbb{T}[(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2]}$ as

$$\begin{aligned}\frac{(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2}{\mathbb{T}[(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2]}=&w_{\varepsilon,t}^2+2\varepsilon w_{\varepsilon,t}\tilde{U}_1-\varepsilon V_1 w_{\varepsilon,t}^2+2w_{\varepsilon,t}\phi-\frac{\int_{-\infty}^{\infty}w\phi}{\xi}w_{\varepsilon,t}^2+\varepsilon^2\tilde{U}_1^2\\&+\varepsilon^2 w_{\varepsilon,t}^2V_1^2-2\varepsilon^2 w_{\varepsilon,t}\tilde{U}_1V_1-\varepsilon^2\Theta_2 w_{\varepsilon,t}^2+O(\varepsilon^{2+\delta}),\end{aligned}\tag{5.9}$$

where

$$\Theta_2=\frac{\varepsilon^2}{2\xi}\int_{-\infty}^{\infty}U_1^2+\frac{\varepsilon^2}{\xi}\int_{-\infty}^{y}(2G_1^+wU_1+G_2^+w^2)+\frac{\varepsilon^2}{\xi}\int_{y}^{\infty}(2G_1^-wU_1+G_2^-w^2).$$

Substituting (5.9) into $\mathbb{M}_\varepsilon^1$, we obtain

$$\|\mathbb{M}_\varepsilon^1[\phi]\|_*\leq C(\varepsilon^2+\|\phi\|_*^2),\tag{5.10}$$

where we have used $U_1$ and $w$ are exponential decay. For the another two terms, it is not difficult to find that

$$\|\mathbb{M}_\varepsilon^2[\phi]\|_*+\|\mathbb{M}_\varepsilon^3[\phi]\|_*\leq C\varepsilon^2.\tag{5.11}$$

Combining (5.10) and (5.11), we get $\|\mathbb{M}_\varepsilon[\phi]\|_*\leq C(\|\phi\|_*^2+\varepsilon^2)$.

Set $\mathfrak{B}=\{\phi\in\mathcal{H}\mid\|\phi\|_*<C\varepsilon^\sigma\}$. Fix $\phi\in\mathfrak{B}$ and let $\mathcal{A}_\varepsilon$ be the unique map $h\to\phi$ given by Proposition 3.3. Defining

$$\mathcal{G}_\varepsilon=\mathcal{A}_\varepsilon(\mathbb{E}_\varepsilon+\mathbb{M}_\varepsilon[\phi]).$$

We now show that $\mathcal{G}_\varepsilon$ is a contraction map. In fact, by Proposition 3.3, we have

$$\|\mathcal{G}_\varepsilon[\phi]\|_*\leq C\|\mathbb{E}_\varepsilon+\mathbb{M}_\varepsilon[\phi]\|_*\leq C\varepsilon^2+\varepsilon^{2\sigma}\leq C\varepsilon^2,\tag{5.12}$$

since $1<\sigma<2$, and hence $\mathcal{G}_\varepsilon[\phi]\in\mathfrak{B}$. Moreover, we also have

$$\begin{aligned}\|\mathcal{G}_\varepsilon[\phi_1]-\mathcal{G}_\varepsilon[\phi_2]\|_*\leq&C\|\mathbb{M}_\varepsilon[\phi_1]-\mathbb{M}_\varepsilon[\phi_2]\|\\\leq&C(\varepsilon+\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*.\end{aligned}\tag{5.13}$$

Eq. (5.12) and (5.13) show that the map $\mathcal{G}_\varepsilon$ is a contraction map from $\mathfrak{B}$ to $\mathfrak{B}$. By the contraction mapping theorem, (5.4) has a unique solution $\phi\in\mathfrak{B}$, called $\phi_{\varepsilon,t}$.

The continuity of $(\phi_{\varepsilon,t},\beta_\varepsilon(t))$ follows from the uniqueness of $(\phi_{\varepsilon,t},\beta_\varepsilon(t))$. □

## 6. The reduced problem

In this section we solve the reduced problem and prove our main result. In particular, we obtain that

**Proposition 6.1.** *For $\varepsilon$ sufficiently small, $\beta_\varepsilon(t)$ is continuous in $t$ and we have*

$$\beta_\varepsilon(t)=\frac{b_0}{t}\varepsilon^2(\frac{309}{700}-\frac{3}{5}ta-\frac{3}{10}a)+o(\varepsilon^{2+\delta}),\tag{6.1}$$

*where $b_0\neq0$ is some generic constant and $\delta\in(0,\sigma)$.*

From Proposition 6.1, we can finish the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Since $\varepsilon^{-2}b_0^{-1}\beta_\varepsilon(t) = \frac{1}{t}(\frac{309}{700} - \frac{3}{5}ta - \frac{3}{10}a) + o(\varepsilon^\delta)$. On the other hand, we have $a = \frac{e^{-2t}}{\varepsilon}$. Then we can write

$$b_0^{-1}\varepsilon^{-2}\beta_\varepsilon(t) = \frac{1}{t}(\frac{309}{700} - \frac{3}{10}\frac{(2t+1)e^{-2t}}{\varepsilon}) + o(\varepsilon^\delta). \tag{6.2}$$

For $\varepsilon$ sufficiently small. From Section 3, we have $1 < t < \frac{1}{\varepsilon^2}$. If we choose $t = 2$, we can make the right hand side of (6.2) negative. While if we choose $t = -2\ln(\varepsilon) < \frac{1}{\varepsilon^2}$ we can make the right hand side of (6.2) positive. By the continuity of $\beta_\varepsilon(t)$ and the mean value theorem, a zero of $\beta_\varepsilon$, denoted by $t_\varepsilon$, is thus guaranteed, which produces a solution $u_\varepsilon = w_{\varepsilon,t} + \varepsilon\tilde{U}_1 + \phi_{\varepsilon,t}$ to (5.1) and (5.2). It is easy to verify that $u_\varepsilon$ satisfies all the properties of Theorem 1.1. □

We now prove Proposition 6.1. Observing that $\phi_{\varepsilon,t}$ satisfies (5.4). Multiplying Eq.(5.4) by $w'_{\varepsilon,t}$ and integrating by over $\mathbb{I}_{\varepsilon,t}$, we obtain

$$\beta_\varepsilon(t)\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t}w'_{\varepsilon,t} = \int_{\mathbb{I}_{\varepsilon,t}} \mathcal{L}_{\varepsilon,t}[\phi_{\varepsilon,t}]w'_{\varepsilon,t} + \int_{\mathbb{I}_{\varepsilon,t}}(-\mathbb{E}_\varepsilon w'_{\varepsilon,t}) + \int_{\mathbb{I}_{\varepsilon,t}}(-\mathbb{M}_\varepsilon[\phi_{\varepsilon,t}]w'_{\varepsilon,t}). \tag{6.3}$$

The left hand side of (6.3) can be computed as:

$$\beta_\varepsilon(t)\int_{\mathbb{I}_{\varepsilon,t}} Z_{\varepsilon,t}w'_{\varepsilon,t} = -2\beta_\varepsilon(t)\int_{\mathbb{R}}(w(w')^2) + O(\varepsilon\beta_\varepsilon(t)). \tag{6.4}$$

We estimate each term on the right hand side of (6.3). For the first term, we use integration by parts:

$$\begin{aligned}\int_{\mathbb{I}_{\varepsilon,t}} \mathcal{L}_{\varepsilon,t}[\phi_{\varepsilon,t}]w'_{\varepsilon,t} &= \int_{\mathbb{I}_{\varepsilon,t}} [\phi''_{\varepsilon,t} - \phi_{\varepsilon,t} + 2w_{\varepsilon,t}\phi_{\varepsilon,t}]w'_{\varepsilon,t} + O(\varepsilon\|\phi_{\varepsilon,t}\|_*)\\ &= \int_{\mathbb{I}_{\varepsilon,t}} [w'''_{\varepsilon,t} - w'_{\varepsilon,t} + 2w_{\varepsilon,t}w'_{\varepsilon,t}]\phi_{\varepsilon,t} = O(\varepsilon^{1+\sigma}).\end{aligned} \tag{6.5}$$

The second term in (6.3) gives, using (5.8),

$$\int_{\mathbb{I}_{\varepsilon,r_0}} \mathbb{E}_\varepsilon\tilde{U}'_0 = O(\varepsilon^3). \tag{6.6}$$

It remains to compute the third term on the right hand side of (6.3), by (5.9),

$$\begin{aligned}\frac{(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2}{\mathbb{T}[(w_{\varepsilon,t}+\varepsilon\tilde{U}_1+\phi)^2]} =& w^2_{\varepsilon,t} + 2\varepsilon w_{\varepsilon,t}\tilde{U}_1 - \varepsilon V_1 w^2_{\varepsilon,t} + 2w_{\varepsilon,t}\phi - \frac{\int_{-\infty}^\infty w\phi}{\xi}w^2_{\varepsilon,t} + \varepsilon^2\tilde{U}_1^2\\ &+ \varepsilon^2 w^2_{\varepsilon,t}V_1^2 - 2\varepsilon^2 w_{\varepsilon,t}\tilde{U}_1V_1 - \varepsilon^2\Theta_2 w^2_{\varepsilon,t} + O(\varepsilon^{2+\delta}),\end{aligned} \tag{6.7}$$

we can write $\mathbb{M}_\varepsilon$ as

$$\begin{aligned}\mathbb{M}_\varepsilon[\phi] =& \varepsilon^2\Big[-\frac{2}{t}\tilde{U}'_1 + \frac{2y}{t^2}w'_{\varepsilon,t} + w^2_{\varepsilon,t}\Theta_2 + 2w_{\varepsilon,t}\tilde{U}_1V_1 - \tilde{U}_1^2 - w^2_{\varepsilon,t}V_1^2\Big]\\ &+ (\frac{2\varepsilon}{t} - \frac{2\varepsilon}{t+\varepsilon y} - \frac{2y\varepsilon^2}{t^2})w'_{\varepsilon,t} + (\frac{2\varepsilon^2}{t} - \frac{2\varepsilon^2}{t+\varepsilon y})\tilde{U}'_1 + o(\varepsilon^{2+\sigma})\\ =& \varepsilon^2\Big[-\frac{2}{t}U'_1 + \frac{2y}{t^2}w' + w^2\Theta_2 + 2wU_1V_1 - U_1^2 - w^2V_1^2\Big]\\ &+ o(\varepsilon^{2+\delta}),\end{aligned} \tag{6.8}$$

where we used that fact that $w$, $U_1$ are exponential decay.

Next, we consider the term

$$\Pi = -\frac{2}{t}U_1' + \frac{2y}{t^2}w' + w^2\Theta_2 + 2wU_1V_1 - U_1^2 - w^2V_1^2.$$

Specifically, we need to compute the inner product of $\Pi$ and $w'_{\varepsilon,t}$, and we find

$$\int_{-\infty}^{\infty} \Pi w'_{\varepsilon,t} = \int_{-\infty}^{\infty} \Pi w' + O(\varepsilon^3).$$

Therefore, it is only necessary to compute $\int_{-\infty}^{\infty} \Pi w'$. From Appendix A, we see

$$\Theta_2 = V_2 - \frac{1}{\xi}\int_{-\infty}^{\infty} wU_2,$$

where $U_2, V_2$ represent the $\varepsilon^2-$order term in the expansion of $u, v$ respectively. Then we find

$$\int_{-\infty}^{\infty} \Pi w' = \int_{-\infty}^{\infty}\left[\frac{2}{t}U_1(w-w^2) - \frac{1}{3}w^3V_2' + 2ww'U_1V_1 - U_1^2w' - w^2w'V_1^2\right], \quad (6.9)$$

where we used the fact

$$\int_{-\infty}^{\infty} yw'^2 = 0,\ \int_{-\infty}^{\infty} w^2w' = 0,\ w'' - w + w^2 = 0.$$

It is to check (6.9) is just the following one

$$\frac{2}{t}(I_1 - I_2) - \frac{1}{3}I_3 + 2I_4 - I_5 - I_6,$$

where

$$\begin{aligned}
I_1 &= \int_0^{\infty} U_{1\text{even}}w,\ I_2 = \int_0^{\infty} U_{1\text{even}}w^2,\ I_3 = \int_0^{\infty} w^3(V_2')_{\text{even}},\\
I_4 &= \frac{1}{2}\int_{-\infty}^{\infty} ww'U_1V_1 = \int_{-\infty}^{\infty} ww'[U_{1\text{even}}V_{1\text{odd}} + V_{1\text{even}}U_{1\text{odd}}],\\
I_5 &= \frac{1}{2}\int_{-\infty}^{\infty} w'U_1^2 = 2\int_0^{\infty} w'(U_{1\text{even}}U_{1\text{odd}}),\\
I_6 &= \frac{1}{2}\int_{-\infty}^{\infty} w^2w'V_1^2 = 2\int_0^{\infty} w^2w'(V_{1\text{even}}V_{1\text{odd}}),
\end{aligned}$$

where $U_{1\text{even}}$ stands for the even function part of $U_1$, $U_{1\text{odd}}$ stands for the odd function part of $U_1$(same notation for $U_0,\ U_2,\ V_0,\ V_1,\ V_2$).

By (7.9), we have

$$I_1 - I_2 = \int_0^{\infty}(C_0w - \phi_1)(w - w^2) = -\frac{3}{5}C_0 + \int_0^{\infty}\phi_1w^2 - \int_0^{\infty}\phi_1w \quad (6.10)$$

and

$$\begin{aligned}
I_3 =& \int_{-\infty}^{\infty} w^3\Big(a - \frac{C_0}{t} + \frac{1}{t}\Big(2\rho(y) + 3\frac{\int_0^{\infty} zw^2}{\int_0^{\infty} w^2} - 2\frac{\int_0^{y} zw^2}{\int_0^{\infty} w^2} - 2\frac{w^2}{\int_0^{\infty} w^2}\Big)\Big)\\
=&\Big(a - \frac{C_0}{t} + \frac{1}{t}\int_0^{\infty} zw^2\Big)\int_{-\infty}^{\infty} w^3 - \frac{2}{3t}\int_0^{\infty} w^3\Big(\int_0^{y} zw^2\mathrm{d}z\Big)\mathrm{d}y\\
&- \frac{2}{3t}\int_0^{\infty} w^5 + \frac{2}{t}\int_0^{\infty} w^3\rho, \quad (6.11)
\end{aligned}$$

where $\rho(y), \phi_1, C_0$ will be given in Appendix A and we have used $\int_0^\infty w^2 = 3$ (this identity will be given in Appendix B). Thus, we obtain

$$I_1 - I_2 - \frac{t}{6}I_3 = \int_0^\infty \phi_1 w^2 - \int_0^\infty \phi_1 w - \frac{3}{5}(ta + \int_0^\infty zw^2) + \frac{1}{9}\int_0^\infty w^3(\int_0^y zw^2 \mathrm{d}z)\mathrm{d}y + \frac{1}{9}\int_0^\infty w^5 - \frac{1}{3}\int_0^\infty w^3\rho. \tag{6.12}$$

Then, in Appendix B, we show

$$I_1 - I_2 - \frac{t}{6}I_3 = -\frac{57}{100} - \frac{3}{5}ta. \tag{6.13}$$

On the other hand, we have

$$I_4 = -\frac{2}{t}C_0\int_0^\infty ww'(w' + yw) + \frac{1}{t}\int_0^\infty ww'[y\phi_1 + (2w' + yw)\rho],$$

$$I_5 = -\frac{2}{t}\int_0^\infty w'(C_0 w - \phi_1)(2w' + yw), \ I_6 = -\frac{2}{t}\int_0^\infty yww'(C_0 - \rho).$$

Hence,

$$-\frac{t}{2}(I_5 + I_6 - 2I_4) = -\int_0^\infty 2w'^2\phi_1 + \int_0^\infty ww'(2w' + yw - y)\rho + C_0\int_0^\infty yww'(1 - w). \tag{6.14}$$

Further, we will show in Appendix B

$$-\frac{t}{2}(I_5 + I_6 - 2I_4) = \frac{177}{175} - \frac{3}{10}a. \tag{6.15}$$

Combining (6.7)-(6.15), we get

$$\int_{-\infty}^\infty \mathbb{M}_{\varepsilon,t}w'_{\varepsilon,t} = \frac{2}{t}\varepsilon^2(\frac{309}{700} - \frac{3}{5}ta - \frac{3}{10}a) + o(\varepsilon^{2+\delta}). \tag{6.16}$$

By (6.3)-(6.6) and (6.16), we have

$$\beta_\varepsilon(t) = \frac{b_0}{t}\varepsilon^2(\frac{309}{700} - \frac{3}{5}ta - \frac{3}{10}a) + o(\varepsilon^{2+\delta}). \tag{6.17}$$

Therefore, we get the proposition proved. □

## 7. Appendix A

In this appendix, we compute the explicit expressions of the first and second approximation of $u$ and $v$. Let $U(y) = u(r)$, $V(y) = v(r)$, $y = \frac{r-t}{\varepsilon}$. We formally write $U, V$ into the following

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots; \quad V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \cdots.$$

By (2.3), we have

$$V(y) \sim \frac{1}{\xi}\int_{-\infty}^\infty G(r_0 + \varepsilon y, r_0 + \varepsilon z)U^2(z)\mathrm{d}z. \tag{7.1}$$

At leading order, we then get

$$V_0 = \frac{G_0}{\xi}\int_{-\infty}^{\infty} U_0^2; \quad U_0'' - U_0 + \frac{U_0^2}{V_0} = 0,$$

where we used the expansion of $G$ in (4.4). We now choose the constant $\xi$ so that $V_0 = 1$ and therefore $U_0 = w$; that is

$$\xi = \int_0^{\infty} w(y)^2 dy; \;\; V_0 = 1; \;\; U_0 = w.$$

Next, after computing all the terms up to $O(\varepsilon^2)$ in (2.1), we obtain

$$\begin{aligned}
&L_0 U_1 + \frac{2}{t}U_0' - U_0^2 V_1 = 0,\\
&L_0 U_2 + \frac{2}{t}U_1' - \frac{2y}{r_0^2}U_0' - U_0^2 V_2 - 2U_0U_1V_1 + U_1^2 + U_0^2 V_1^2 = 0,\\
&V_1'' + \frac{1}{\xi}U_0^2 = 0,\\
&V_2'' + \frac{2}{t}V_1' - 1 + \frac{2}{\xi}U_0U_1 = 0,
\end{aligned} \tag{7.2}$$

where the operator $L_0$ is defined as $L_0\psi = (\partial^2 - 1 + 2U_0)\psi$. Multiplying the first equation in (7.2) by $U_0'$, we get

$$\frac{2}{t}\int_{-\infty}^{\infty} U_0'^2 + \frac{1}{3}\int_{-\infty}^{\infty} U_0^3 V_1' = 0. \tag{7.3}$$

From (7.1), we have

$$\xi V_1 = 2G_0\int_{-\infty}^{\infty} U_0U_1 \mathrm{d}z + \int_{-\infty}^{y} G_1^+(y,z)U_0^2(z)\mathrm{d}z + \int_{y}^{\infty} G_1^-(y,z)U_0^2(z)\mathrm{d}z. \tag{7.4}$$

Differentiating the both two sides with respect to $y$, we get

$$\begin{aligned}
\xi V_1' &= \int_{-\infty}^{y}(-\frac{1}{2} - \frac{1}{2t})U_0^2(z)\mathrm{d}z + \int_{y}^{\infty}(\frac{1}{2} - \frac{1}{2t})U_0^2(z)\mathrm{d}z\\
&= -\frac{1}{2t}\int_{-\infty}^{\infty} U_0^2(z)\mathrm{d}z + f_1,
\end{aligned}$$

where $f_1$ is an odd function of $y$. In addition, we can write

$$V_1' = -\frac{1}{t} + f_2, \;\; f_2 \text{ is an odd function of } y.$$

Then, the left hand side of (7.3) becomes

$$\frac{2}{t}\int_{-\infty}^{\infty} U_0'^2 - \frac{1}{3t}\int_{-\infty}^{\infty} U_0^3 = 0.$$

This expression is identically zero, since $\int_{-\infty}^{\infty} U_0'^2 = \frac{6}{5}$ and $\int_{-\infty}^{\infty} U_0^3 = \frac{36}{5}$. These two identities can be directly proven by using $w(y) = \frac{3}{2}\mathrm{sech}^2(\frac{y}{2})$.

Next we consider the function $V_2$. From (7.1), we have

$$\xi V_2 = \int_{-\infty}^{\infty} G_2U_0^2 + \int_{-\infty}^{\infty} 2G_1U_0U_1 + \int_{-\infty}^{\infty} G_0(U_1^2 + 2U_0U_2). \tag{7.5}$$

Actually we only care about the even part of $V_2'$. We have

$$\begin{aligned}\xi V_2' =&[G_2^+(y,y) - G_2^-(y,y)]U_0^2 + \int_{-\infty}^{y} G_{2y}^+ U_0^2 + \int_{y}^{\infty} G_{2y}^- U_0^2 \\ &+ 2[G_1^+(y,y) - G_1^-(y,y)]U_0U_1 + \int_{-\infty}^{y} 2G_{1y}^+ U_0U_1 + \int_{y}^{\infty} 2G_{1y}^- U_0U_1, \end{aligned} \quad (7.6)$$

where $G_{2y}^+ = \frac{\partial G_2^+}{\partial y}$, similar notation will be used for $G_1^+, G_1^-, G_2^-$ respectively. Since

$$G_2^+(y,y) - G_2^-(y,y) = G_1^+(y,y) - G_1^-(y,y) = 0,$$

(7.6) becomes

$$\xi V_2' = \int_{-\infty}^{y} G_{2y}^+ U_0^2 + \int_{y}^{\infty} G_{2y}^- U_0^2 + \int_{-\infty}^{y} 2G_{1y}^+ U_0U_1 + \int_{y}^{\infty} 2G_{1y}^- U_0U_1. \quad (7.7)$$

Now we evaluate the first two terms on the right hand side of (7.7), keeping only the even terms in $y$. We have

$$G_{2y}^- = \frac{1}{2t^2}(y + (y-z)(t-1)^2) + \frac{a}{2t}(t+1),$$

$$G_{2y}^+ = \frac{1}{2t^2}(y + (y-z)(t+1)^2) + \frac{a}{2t}(t+1),$$

then

$$\int_{-\infty}^{y} G_{2y}^+ U_0^2 + \int_{y}^{\infty} G_{2y}^- U_0^2 = \int_{0}^{y} G_{2y}^+ U_0^2 + \int_{y}^{0} G_{2y}^- U_0^2 + \int_{-\infty}^{0} G_{2y}^+ U_0^2 + \int_{0}^{\infty} G_{2y}^- U_0^2,$$

$$\int_{-\infty}^{0} G_{2y}^+ U_0^2 + \int_{0}^{\infty} G_{2y}^- U_0^2 = a(1 + \frac{1}{t}) \int_{0}^{\infty} U_0^2 + \frac{2}{t} \int_{0}^{\infty} zU_0^2 + f_3,$$

and

$$\int_{0}^{y} G_{2y}^+ U_0^2 + \int_{y}^{0} G_{2y}^- U_0^2 = \frac{2}{t} \int_{0}^{y} (y-z)U_0^2,$$

where $f_3$ is an odd function of $y$. So that

$$\int_{-\infty}^{y} G_{2y}^+ U_0^2 + \int_{y}^{\infty} G_{2y}^- U_0^2 = a(1 + \frac{1}{t}) \int_{0}^{\infty} U_0^2 + \frac{2}{t} \int_{0}^{\infty} zU_0^2 + \frac{2}{t} \int_{0}^{y} (y-z)U_0^2 + f_3.$$

Next we compute the terms involving $U_1$. We can write

$$\begin{aligned}\int_{-\infty}^{y} 2G_{1y}^+ U_0U_1 + \int_{y}^{\infty} 2G_{1y}^- U_0U_1 =& \int_{-\infty}^{0} 2G_{1y}^+ U_0U_1 + \int_{0}^{\infty} 2G_{1y}^- U_0U_1 \\ &+ \int_{0}^{y} 2G_{1y}^+ U_0U_1 + \int_{y}^{0} 2G_{1y}^- U_0U_1. \end{aligned} \quad (7.8)$$

Since $G_{1y}^- = (\frac{1}{2} - \frac{1}{2t})$ and $G_{1y}^+ = (-\frac{1}{2} - \frac{1}{2t})$, we get

$$\begin{aligned}\int_{-\infty}^{0} 2G_{1y}^+ U_0U_1 + \int_{0}^{\infty} 2G_{1y}^- U_0U_1 =& \int_{-\infty}^{0} (-1 - \frac{1}{t}) U_0U_1 + \int_{0}^{\infty} (1 - \frac{1}{t}) U_0U_1 \\ =& -\frac{2}{t} \int_{0}^{\infty} U_0 U_{1\text{even}} + 2 \int_{0}^{\infty} U_0 U_{1\text{odd}},\end{aligned}$$

and

$$\int_0^y 2G_{1y}^+ U_0 U_1 + \int_y^0 2G_{1y}^- U_0 U_1 = \int_0^y (\frac{1}{t} - 1) U_0 U_{1\text{odd}} - \int_0^y (\frac{1}{t} + 1) U_0 U_{1\text{odd}} + f_4$$
$$= -2 \int_0^y U_0 U_{1\text{odd}} + f_4,$$

where $f_4$ is a odd function of $y$. Let's define $\rho(y) := \frac{\int_0^y (y-z) U_0^2 \mathrm{d}z}{\xi}$. Then we have

$$(V_2')_{\text{even}} = a(1 + \frac{1}{t}) + \frac{2}{t\xi} \int_0^\infty z U_0^2 + \frac{2}{t} \rho(y) - \frac{2}{t\xi} \int_0^\infty U_0 U_{1\text{even}}$$
$$+ \frac{2}{\xi} \int_0^\infty U_0 U_{1\text{odd}} - \frac{2}{\xi} \int_0^y U_0 U_{1\text{odd}}.$$

Next, we compute $U_1, V_1$ explicitly. We have

$$\xi V_1 = \int_{-\infty}^\infty U_0(z) U_1(z) \mathrm{d}z + \int_0^\infty G_1^- U_0^2 + \int_{-\infty}^0 G_1^+ U_0^2 + \int_y^0 G_1^- U_0^2 + \int_0^y G_1^+ U_0^2,$$
$$\int_{-\infty}^0 G_1^+ U_0^2 + \int_0^\infty G_1^- U_0^2 = -\frac{y}{t} \int_0^\infty U_0^2 + \int_0^\infty (-z U_0^2 - a U_0^2),$$
$$\int_0^y G_1^+ U_0^2 + \int_y^0 G_1^- U_0^2 = -\int_0^y (y - z) U_0^2.$$

So that

$$V_1 = -\frac{y}{t} - a - \frac{1}{\xi} \int_0^\infty z U_0^2 - \rho(y) + \frac{1}{\xi} \int_{-\infty}^\infty U_0(z) U_1(z) \mathrm{d}z.$$

Separating the odd and even part of $V_1$ and $U_1$, we find

$$V_{1\text{odd}} = -\frac{y}{t},\ V_{1\text{even}} = -a - \frac{1}{\xi} \int_0^\infty z U_0^2 - \rho(y) + \frac{2}{\xi} \int_0^\infty U_0 U_{1\text{even}},$$

$U_{1\text{odd}}$ and $U_{1\text{even}}$ satisfies

$$L_0 U_{1\text{odd}} + \frac{2}{t} U_0' + \frac{y}{t} U_0^2 = 0,$$
$$L_0 U_{1\text{even}} - U_0^2 (-a - \frac{1}{\xi} \int_0^\infty z U_0^2 - \rho(y) + \frac{2}{\xi} \int_0^\infty U_0 U_{1\text{even}}) = 0.$$

Therefore

$$U_{1\text{even}} = \left( -a - \frac{1}{\xi} \int_0^\infty z U_0^2 + \frac{2}{\xi} \int_0^\infty U_0 U_{1\text{even}} \right) U_0 - L_0^{-1} (U_0^2 \rho),$$
$$U_{1\text{odd}} = -\frac{y}{t} U_0 - B U_0',$$

where $B$ is defined later. Now we can get

$$\int_0^\infty U_0 U_{1\text{even}} = -a \int_0^\infty U_0^2 - \int_0^\infty z U_0^2 + 2 \int_0^\infty U_0 U_{1\text{even}} - \int_0^\infty U_0 L_0^{-1} (U_0^2 \rho),$$

so that

$$\int_0^\infty U_0 U_{1\text{even}} = a \int_0^\infty U_0^2 + \int_0^\infty z U_0^2 + \int_0^\infty U_0 L_0^{-1} (U_0^2 \rho).$$

As a consequence,

$$U_{1\text{even}} = \left( a + \frac{1}{\xi} \int_0^\infty z U_0^2 + \frac{2}{\xi} \int_0^\infty U_0 L_0^{-1} (U_0^2 \rho) \right) U_0 - L_0^{-1} (U_0^2 \rho).$$

To get the constant $B$, we impose $U_1'(0) = 0$. We have

$$-U_1'(0) = \frac{1}{t}U_0(0) + BU_0''(0) = \frac{1}{t}\frac{3}{2} + B(\frac{3}{2} - \frac{9}{4}) = 0,$$

so that $B = \frac{2}{t}$. Then we obtain

$$U_{1\text{odd}} = -\frac{1}{t}(2U_0' + yU_0).$$

Combining the formula of $U_{1\text{even}}$ and $V_{1\text{even}}$, we get

$$V_{1\text{even}} = a + \frac{1}{\xi}\int_0^\infty zU_0^2 - \rho(y) + \frac{2}{\xi}\int_0^\infty U_0L_0^{-1}(U_0^2\rho).$$

Next we simplify $(V_2')_{\text{even}}$ as follows. First, we have

$$\int_0^\infty U_0U_{1\text{even}} = a\int_0^\infty U_0^2 + \int_0^\infty zU_0^2 + \int_0^\infty U_0L_0^{-1}(U_0^2\rho),$$

$$\int_0^y U_0U_{1\text{odd}} = -\frac{1}{t}(U_0^2 - \frac{9}{4} + \int_0^y zU_0^2),$$

$$\int_0^\infty U_0U_{1\text{odd}} = -\frac{1}{t}(-\frac{9}{4} + \int_0^\infty zU_0^2).$$

Then, we define

$$\rho(y) := \frac{1}{\xi}\int_0^y (y-z)U_0^2,\ \phi_1 = L_0^{-1}(U_0^2\rho),\ C_0 := a + \frac{1}{\xi}\int_0^\infty zU_0^2 + \frac{2}{\xi}\int_0^\infty U_0\phi_1.$$

Summarizing our results, which are the following

$$\begin{aligned}
&U_{1\text{even}} = C_0U_0 - \phi_1, \quad U_{1\text{odd}} = -\frac{1}{t}(2U_0' + yU_0),\\
&V_{1\text{even}} = C_0 - \rho(y), \quad V_{1\text{odd}} = -\frac{y}{t},\\
&V_{2y,even} = a - \frac{C_0}{t} + \frac{1}{t}\left(2\rho(y) + \frac{3}{\xi}\int_0^\infty zU_0^2 - \frac{2}{\xi}\int_0^y zU_0^2 - \frac{2}{\xi}U_0^2\right). \qquad (7.9)
\end{aligned}$$

## 8. Appendix B

In this appendix we compute explicitly the expressions (6.13) and (6.15). We claim

$$I_1 - I_2 - \frac{t}{6}I_3 = -\frac{57}{100} - \frac{3}{5}ta, \qquad (8.1)$$

$$-\frac{t}{2}(I_5 + I_6 - 2I_4) = \frac{177}{175} - \frac{3}{10}a. \qquad (8.2)$$

Let's define

$$J(p) := \int_0^\infty w^p\rho, \quad K(p) := \int_{-\infty}^\infty w^p\rho_y, \quad M(p) := \int_0^\infty w^p,$$

where $\rho = \frac{1}{3}\int_0^y (z-y)w^2(z)\mathrm{d}z$. Using the equations

$$w'' - w + w^2 = 0, \quad w'^2 - w^2 + \frac{2}{3}w^3 = 0, \quad \rho_{yy} = \frac{1}{3}w^2,$$

we derive the following recursion identities

$$J(p)=\frac{3(p-1)}{2p-1}J(p-1)-\frac{1}{(p-1)(2p-1)}M(p+1),\quad p>1, \tag{8.3}$$

$$K(p)=\frac{3(p-1)}{2p-1}K(p-1)-\frac{2p}{(p+1)(p-1)(2p-1)}M(p+1),\quad p>1, \tag{8.4}$$

$$M(p)=\frac{3(p-1)}{2p-1}M(p-1),\quad p>1. \tag{8.5}$$

The proof of (8.3)-(8.5) are directly and we omit the details. Using (8.3)-(8.5), we obtain the following table

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $M(i)$ | 3 | 3 | $\frac{18}{5}$ | $\frac{162}{35}$ | $\frac{216}{35}$ |
| $K(i)$ | $\frac{7}{2}$ | $\frac{19}{10}$ | $\frac{111}{70}$ | | |
| $J(i)$ | $\frac{13}{2}-6\ln(2)$ | $\frac{53}{10}-6\ln(2)$ | $\frac{1032}{175}-\frac{36}{5}\ln(2)$ | $\frac{8928}{1225}-\frac{324}{35}\ln(2)$ | |

In addition we evaluate

$$\int_0^\infty zw^2=-\frac{3}{2}+6\ln(2),\quad \int_0^\infty w^3(\int_0^y zw^2\mathrm{d}z)=-\frac{4527}{350}+\frac{108}{5}\ln(2). \tag{8.6}$$

We now compute

$$\int_0^\infty \phi_1 w^2=\int_0^\infty w^3\rho=J(3)=\frac{1032}{175}-\frac{36}{5}\ln(2) \tag{8.7}$$

$$\begin{aligned}\int_0^\infty \phi_1 w&=\int_0^\infty w^2\rho(w+\frac{1}{2}yw')=\int_0^\infty w^3\rho+\frac{1}{6}\int_0^\infty y(w^3)'\rho\\ &=\int_0^\infty \frac{5}{6}w^3\rho-\int_0^\infty\frac{1}{6}w^3y\rho'=\frac{5}{6}J(3)-\frac{1}{6}K(3)\\ &=\frac{93}{20}-6\ln(2).\end{aligned} \tag{8.8}$$

To evaluate $\int_0^\infty w'^2\phi_1$, we note

$$L_0w^2=2w''w+2w'^2-w^2+2w^3=2w'^2+w^2,$$

so that $L_0^{-1}w'^2=\frac{1}{2}(w^2-w)$. Thus

$$-\int_0^\infty 2w'^2\phi_1=-\int_0^\infty(w^2-w)\rho=\int_0^\infty w''\rho=\frac{1}{3}\int_0^\infty w^3=\frac{6}{5}. \tag{8.9}$$

Finally, we get

$$C_0=a+\frac{1}{3}\int_0^\infty zw^2+\frac{2}{3}\int_0^\infty w\phi_1=a+\frac{13}{5}-2\ln(2), \tag{8.10}$$

$$\int_0^\infty yww'(1-w)=\int_0^\infty yw''w'=-\frac{1}{2}\int_0^\infty w'^2=-\frac{3}{10}, \tag{8.11}$$

$$
\begin{aligned}
\int_0^\infty ww'(2w'+yw-y)\rho =& \int_0^\infty 2w'^2 w\rho - yw'w''\rho \\
=& \int_0^\infty (2(w^2-\frac{2}{3}w^3)w\rho + \frac{1}{2}w'^2(\rho + y\rho_y)) \\
=& \int_0^\infty (2w^3 - \frac{4}{3}w^4)\rho + \int_0^\infty (\frac{1}{2}w^2 - \frac{1}{3}w^3)(\rho + y\rho_y) \\
=& \frac{1}{2}J(2) + \frac{5}{3}J(3) - \frac{4}{3}J(4) + \frac{1}{2}K(2) - \frac{1}{3}K(3) \\
=& \frac{7797}{2450} - \frac{93}{35}\ln(2). \qquad (8.12)
\end{aligned}
$$

Then, using (8.3)-(8.12), we get the claim proved.

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