On Ring-like Solutions For the Gray-Scott Model: Existence, Instability and Self-Replicating Rings

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We consider the following Gray-Scott model in $B_R(0)=\{x:|x|<R\}\subset\mathbb{R}^N, N=2,3$:

$$\begin{cases} v_t=\varepsilon^2\Delta v-v+Av^2u & \text{in } B_R(0),\\ \tau u_t=\Delta u+1-u-v^2u & \text{in } B_R(0),\\ u,v>0;\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0 & \text{on } \partial B_R(0)\end{cases}$$

where $\varepsilon>0$ is a small parameter. We assume that $A=\hat{A}\varepsilon^{\frac{1}{2}}$. For each $\hat{A}<+\infty$ and $R<\infty$, we construct ring-like solutions which concentrate on an $(N-1)-$dimensional sphere for the stationary system for all sufficiently small ε. More precisely, it is proved the above problem has a radially symmetric steady state solution $(v_{\varepsilon,R},u_{\varepsilon,R})$ with the property that $v_{\varepsilon,R}(r)\to 0$ in $\mathbb{R}^N\backslash\{r\neq r_0\}$ for some $r_0\in(0,R)$. Then we show that for $N=2$ such solutions are unstable with respect to angular fluctuations of the type $\Phi(r)e^{\sqrt{-1}m\theta}$ for some m. A relation between $\hat{A}$ and the minimal mode m is given. Similar results are also obtained when $\Omega=\mathbb{R}^N$ or $\Omega=B_{R_2}(0)\backslash B_{R_1}(0)$ or $\Omega=\mathbb{R}^N\backslash B_R(0)$.

Keywords: Ring-like Solutions, Gray-Scott Model, Pattern Formation, Singular Perturbations.

1 Introduction and Main Results

The Gray-Scott model [**16**], [**17**] models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the reaction. In dimensionless units it can be written as

$$\begin{cases} V_t=D_V\Delta V-(F+k)V+UV^2 & \text{in } \Omega,\\ U_t=D_U\Delta U+F(1-U)-UV^2 & \text{in } \Omega,\\ \frac{\partial U}{\partial\nu}=\frac{\partial V}{\partial\nu}=0 \quad \text{on } \partial\Omega,\end{cases}\tag{1.1}$$

where the unknowns $U=U(x,t)$ and $V=V(x,t)$ represent the concentrations of the two biochemicals at a point $x\in\Omega\subset\mathbb{R}^N, N\leq 3$ and at a time $t>0$, respectively; $\Delta:=\sum_{j=1}^N\frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; Ω is a bounded and smooth domain in $\mathbb{R}^N$; $\nu(x)$ is the outer normal at $x\in\partial\Omega$; D_U,D_V are the diffusion coefficients of U and V respectively. F denotes the rate at which U is fed from the reservoir into the reactor, and k is a reaction-time constant.

For various ranges of these parameters, (1.1) are known to admit a rich solution structure involving pulses or spots, rings, stripes, traveling waves, self-replication spots, and spatio-temporal chaos. See [**11**], [**38**], [**40**], [**41**], [**25**], [**26**], [**34**], [**35**], [**36**], [**37**] for numerical simulations and experimental observations.

In one-dimesnional case, the first rigorous result in constructing single pulse solution is given in [**11**] for (1.1) in the case $D_U = 1, D_V = \delta^2 \ll 1$. In [**11**], it is assumed that $F \sim \delta^2, F + k \sim \delta^{2\alpha/3}$, where $\alpha \in [0, \frac{3}{2})$. Later the stability of single and multi-pulse solutions in 1-D are obtained in [**10**]. Periodic patterns are constructed in [**13**]. In [**40**], a formal matched asymptotic analysis is used to study the dynamics of self-replicating pulses. The case $D_U = D_V$ and the existence and stability of single and multiple pulse solutions are established in [**19**] and [**20**]. In [**34**], a skeleton structure of self-replicating dynamics is proposed, while in [**35**], [**36**], [**37**], spatio-temporal chaos is observed and analyzed. In all of the above four papers, it is assumed that the diffusivity ratio $D_v/D_u = O(1)$. In such a case, the results are largely numerical. A more detailed analysis is possible when $D_v/D_u \ll O(1)$. In [**23**], the equilibria, Hopf bifurcations, and pulse-splitting dynamics for (1.1) in a finite interval are studied under the small diffusivity ratio assumption. Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can also be found in [**14**], [**29**], [**30**] and the references therein.

In higher-dimensional case, some formal asymptotic analysis on the construction and stability of spotty solution in $\mathbb{R}^2$ and $\mathbb{R}^3$ is given in [**29**] and [**30**]. In [**44**], the second author studied (1.1) in a bounded domain for the shadow system case which can be reduced to a single equation. For spotty solutions for single equations, please see [**5**], [**6**], [**18**],[**47**], [**48**], and the references therein. A good review can be found in [**32**].

The first rigorous result on the existence and stability of spotty solutions in R^2 was given in [**45**]. To state the result, it is important to introduce a suitable scaling.

Let us first transform the system (1.1). We follow the notations in [**29**]. Set

$$\varepsilon^2 = \frac{D_V F}{D_U (F+k)}, \quad A = \frac{\sqrt{F}}{F+k}, \quad \tau = \frac{F+k}{F},$$

$$x = \sqrt{\frac{D_U}{F}}\bar{x}, \quad t = \frac{1}{F+k}\bar{t},$$

$$V(x,t) = \sqrt{F} v(\bar{x}, \bar{t}), \quad U(x,t) = u(\bar{x}, \bar{t}).$$

Let us drop the bar from now on. It is easy to see that (1.1) is equivalent to the following system

$$\begin{cases} v_t = \varepsilon^2 \Delta v - v + A u v^2, \quad \text{in } \Omega, \\ \tau u_t = \Delta u - u v^2 + (1-u), \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Note that there are three parameters (ε, A, τ) in equation (1.2). Throughout this paper, we always assume that

$$0 < \varepsilon \ll 1. \tag{1.3}$$

To study (1.2), we first consider the stationary equation of (1.2):

$$\begin{cases} \varepsilon^2 \Delta v - v + A u v^2 = 0, \quad x \in \Omega, \\ \Delta u - u v^2 + (1-u) = 0, \; x \in \Omega, \\ u > 0, v > 0 \;\text{ in } \Omega, \;\; \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.4}$$

In [**45**], under the condition that

$$\Omega = \mathbb{R}^2, \quad \tau \sim O(1), \quad A \sim \varepsilon(\log \frac{1}{\varepsilon})^{\frac{1}{2}} \tag{1.5}$$

it is proved that Problem (1.2) has two branches of single spotty steady-state solutions in $\mathbb{R}^2$, with one of them being stable and the other one being unstable. The existence and stability of symmetric and asymmetric multiple spotty solutions in a bounded two-dimensional domain are studied in [**49**] and [**50**].

As far as the authors know, there has been no rigorous result in $\mathbb{R}^N, N \geq 3$. Since the physical space is $\mathbb{R}^3$, it is natural to study the Gray-Scott model in $\mathbb{R}^3$.

In this paper, we take a different regime of A: we assume that

$$A = \sqrt{6}\hat{A}\varepsilon^{\frac{1}{2}} \tag{1.6}$$

where $\hat{A}$ is independent of ε. We are concerned with the existence of solutions which concentrates on an $(N-1)$-sphere. This kind of solution is called a ring-like solution. In particular, we can prove the existence of a ring-like solution for *all* $N \geq 2$, which is of independent interest.

A recent work by Morgan and Kaper [**28**] also examines such solutions (a preliminary report of [**28**] was announced in 2001 [**21**]; and was also reported in Morgan's PhD thesis [**27**]). The differences and similarities between [**28**] and this paper are discussed in Section 11.

By suitable scaling, we shall study the existence of ring-like solutions for the following Gray-Scott model in a ball $B_R(0)$:

$$\begin{cases} \varepsilon^2 \Delta v - v + \hat{A}uv^2 = 0, \ \text{ in } \ B_R(0), \\ \Delta u - (6\varepsilon)^{-1}uv^2 + (1-u) = 0, \ \text{ in } \ B_R(0), \\ u > 0, v > 0 \text{ in } \ B_R(0), \\ \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_R(0), \end{cases} \tag{1.7}$$

and the stability of the ring-like solutions for the corresponding Gray-Scott model

$$\begin{cases} v_t = \varepsilon^2 \Delta v - v + \hat{A}uv^2, \ \text{in } B_R(0), \\ \tau u_t = \Delta u - (6\varepsilon)^{-1}uv^2 + (1-u), \ \text{in } B_R(0), \\ u > 0, v > 0 \text{ in } B_R(0), \\ \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_R(0). \end{cases} \tag{1.8}$$

Extensions to $\mathbb{R}^N$ or to an annulus or to the exterior of a ball will be discussed in Section 8.

2 Main Results: Existence and Stability of Ring-like Solutions

We now state our main results.

We first define two functions, to be used later: let $J_1(r)$ be the radially symmetric solution of the following problem

$$J_1^{''} + \frac{N-1}{r}J_1^{'} - J_1 = 0, \quad J_1^{'}(0) = 0, \quad J_1(0) = 1, \quad J_1 > 0. \tag{2.1}$$

The second radially symmetric function, called $J_2(r)$, satisfies

$$J_2^{''} + \frac{N-1}{r}J_2^{'} - J_2 + \delta_0 = 0, \quad J_2 > 0, \quad J_2(+\infty) = 0. \tag{2.2}$$

Here δ_0 is the Dirac measure at 0.

In the case of $N=2$, $J_1(r)=I_0(r)$ and $J_2(r)=\frac{1}{2\pi}K_0(r)$ are the modified Bessel's functions of order 0. (See [4]). In the case of $N=3$, J_1, J_2 can be computed explicitly:

$$J_1=\frac{\sinh r}{r}, \quad J_2(r)=\frac{e^{-r}}{4\pi r}. \tag{2.3}$$

Fix $R>0$. We then define a new $J_{2,R}$

$$J_{2,R}(r)=J_2(r)-\frac{J_2'(R)}{J_1'(R)}J_1(r) \tag{2.4}$$

and a new Green's function $G_R(r;r_0)$

$$G_R''+\frac{N-1}{r}G_R'-G_R+\delta_{r_0}=0, \quad G_R'(R;r_0)=0. \tag{2.5}$$

It is easy to see that

$$G_R(r;r_0)=\frac{1}{J_1'(r_0)J_{2,R}(r_0)-J_1(r_0)J_{2,R}'(r_0)}\begin{cases} J_{2,R}(r_0)J_1(r), & \text{for } r<r_0, \\ J_1(r_0)J_{2,R}(r), & \text{for } r>r_0. \end{cases} \tag{2.6}$$

Fix a $\hat{A}>0$. We suppose the following equation has a unique solution $\xi=\xi(\hat{A},r)$:

$$(1-\xi)\xi=\frac{G_R(r;r)}{\hat{A}^2}, \ 0<\xi<\frac{1}{2}. \tag{2.7}$$

Put

$$M_R(r):=\frac{(N-1)}{r}+\frac{1-\xi}{\xi}\left(\frac{J_1'(r)}{J_1(r)}+\frac{J_{2,R}'(r)}{J_{2,R}(r)}\right), \tag{2.8}$$

where ξ solves (2.7).

Let $w(y)$ be the unique solution for the following ODE:

$$w''-w+w^2=0 \text{ in } \mathbb{R}, \quad w>0, \quad w(0)=\max_{y\in R} w(y), \quad w(y)\to 0 \text{ as } |y|\to\infty. \tag{2.9}$$

In fact, it is easy to see that $w(y)$ can be written explicitly

$$w(y)=\frac{3}{2}\operatorname{sech}^2\left(\frac{y}{2}\right).$$

Then we have the following

Theorem 2.1 *Suppose that $0<R<\infty$. Then for any $\hat{A}<\infty$ and for ε sufficiently small, problem (1.7) has a solution $(v_{\varepsilon,R}, u_{\varepsilon,R})$ with the following properties:*

(1) $v_{\varepsilon,R}, u_{\varepsilon,R}$ are radially symmetric,

(2) $v_{\varepsilon,R}(r)=(1+o(1))\frac{1}{\hat{A}\xi_\varepsilon}w(\frac{r-r_\varepsilon}{\varepsilon})$,

(3) $u_{\varepsilon,R}(r)=1-(1+o(1))\frac{G_R(r;r_\varepsilon)}{\hat{A}^2\xi_\varepsilon}$, where $G_R(r;r_\varepsilon)$ satisfies:

$$G_R''+\frac{N-1}{r}G_R'-G_R+\delta_{r_\varepsilon}=0, \quad G_R'(R;r_\varepsilon)=0, \tag{2.10}$$

where ξ_ε is the root of

$$1-\xi_\varepsilon = \frac{G_R(r_\varepsilon; r_\varepsilon)}{\hat{A}^2\xi_\varepsilon}, \quad 0<\xi_\varepsilon<\frac{1}{2}, \tag{2.11}$$

and $r_\varepsilon \to r_0 \in (0, r_R)$ where $M_R(r_0)=0$ and r_R is such that

$$(J_1 J_{2,R})^{'}(r_R) = 0. \tag{2.12}$$

From Theorem 2.1, for finite R, we see that for each $\hat{A}$, there exists a ring-like solution to (1.7). In fact, it is easy to show that $M_R(r_0)=0$ if and only if the following holds

$$\hat{A}^2 = -G_R(r;r)\left(1-\frac{r}{N-1}\frac{(J_1J_{2,R})'}{J_1J_{2,R}}\right)^2 \frac{N-1}{r}\frac{J_1J_{2,R}}{(J_1J_{2,R})'} \tag{2.13}$$

A graph of r versus $\hat{A}^2$ for $N=2$ and $R=5$ is given in Figure 1.

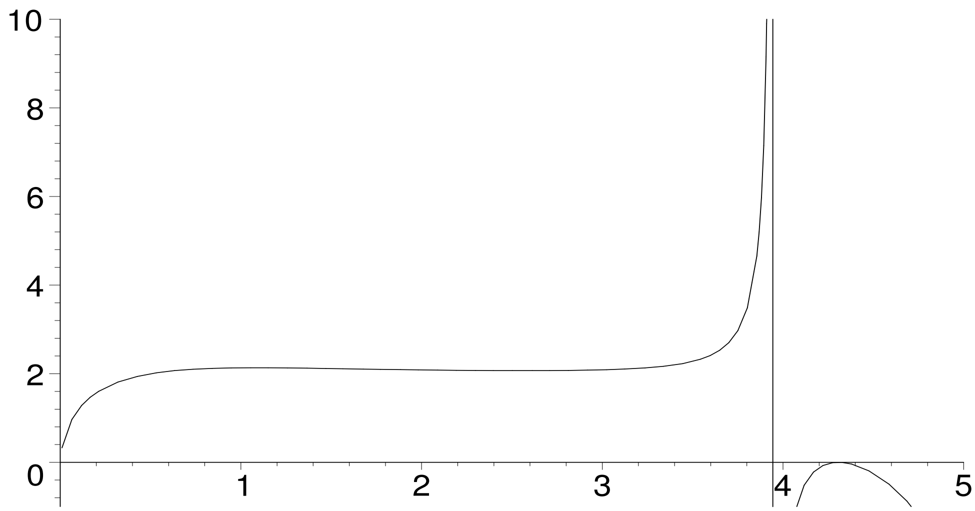

FIGURE 1. The graph of r versus $\hat{A}^2$. Here $N=2,\ R=5$. The singularity occurs at $r_R = 3.94$.

From the graph, we see that $\hat{A}^2$ blows up as $r \to r_R$ where $r_R < R$ satisfies (2.12). In fact, if r_R is a zero root of $(J_1J_{2,R})^{'}$ (see Lemma 3.4 for the existence of r_R), then from (2.13) we see that $\hat{A}^2 \to +\infty$ when $r \to r_R$. (The situation is very different for infinite R. See Section 8.)

A similar existence result for the Gierer-Meinhardt system has also been obtained in [**33**]. Theorem 2.1 is also related to papers [**2**], [**3**] where solutions concentrating on a sphere are constructed for singularly perturbed nonlinear elliptic equations.

Next we study the stability of $(v_{\varepsilon,R},\ u_{\varepsilon,R})$ in $N=2$ with respect to perturbations in the following form:

$$v = v_{\varepsilon,R} + \delta\phi_\varepsilon(r)\cos(m\theta), \quad u = u_{\varepsilon,R} + \delta\psi_\varepsilon(r)\cos(m\theta)$$

where δ is small and $\phi_\varepsilon(r) \sim r^m, \psi_\varepsilon(r) \sim r^m$ for r near 0. That is, we study the eigenvalue problem of the following type

$$\begin{cases} \varepsilon^2\Delta\phi_\varepsilon - \frac{\varepsilon^2 m^2}{r^2}\phi_\varepsilon - \phi_\varepsilon + 2\hat{A}v_{\varepsilon,R}u_{\varepsilon,R}\phi_\varepsilon + \hat{A}v_{\varepsilon,R}^2\psi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon, \\ \Delta\psi_\varepsilon - \frac{m^2}{r^2}\psi_\varepsilon - \psi_\varepsilon - (6\varepsilon)^{-1}2v_{\varepsilon,R}u_{\varepsilon,R}\phi_\varepsilon - (6\varepsilon)^{-1}v_{\varepsilon,R}^2\psi_\varepsilon = \tau\lambda_\varepsilon\psi_\varepsilon, \\ \phi_\varepsilon = \phi_\varepsilon(r), \psi_\varepsilon = \psi_\varepsilon(r), \\ \phi_\varepsilon'(R) = \psi_\varepsilon'(R) = 0. \end{cases} \tag{2.14}$$

We introduce the following important function

$$\rho_{m,R}(r) = \frac{I_m(r)K_m(r) - \frac{K_m'(R)}{I_m'(R)}I_m^2(r)}{I_0(r)K_0(r) - \frac{K_0'(R)}{I_0'(R)}I_0^2(r)}, \tag{2.15}$$

where I_m, K_m are the two modified Bessel's function of order m. See [**4**] for the definitions.

Now we have

Theorem 2.2 *Assume that $N=2$. Let $(v_{\varepsilon,R}, u_{\varepsilon,R})$ be the solution constructed in Theorem 2.1 and suppose $m \ll \frac{1}{\varepsilon}$. If*

$$\rho_{m,R}(r_0) < \frac{\xi}{1-\xi} \tag{2.16}$$

where ξ is given by (2.7), then the problem (2.14) has an eigenvalue with positive real part. If τ is small and

$$\rho_{m,R}(r_0) > \frac{\xi}{1-\xi} \tag{2.17}$$

then all the eigenvalues of the problem (2.14) have negative real parts.

The mode $m=0$ is stable.

Our final theorem shows that when $\hat{A} = O(1)$, the ring solution is always unstable with respect to some wide band of modes m, and that it is stable with respect to very large modes m.

Theorem 2.3 *Assume that $N=2$. Let $(v_{\varepsilon,R}, u_{\varepsilon,R})$ be the solution constructed in Theorem 2.1. If*

$$1 \ll m < \delta_0\frac{1}{\varepsilon} \tag{2.18}$$

for some $\delta_0 < \frac{\sqrt{5}}{2}$, then the problem (2.14) has a positive eigenvalue. On the other hand, if

$$m > \frac{\sqrt{5}}{2}\frac{r_0}{\varepsilon}, \tag{2.19}$$

then the problem (2.14) has no large unstable eigenvalues.

Fix $R=5$. The graph in Figure 2 shows the relation between r and the minimal mode m.

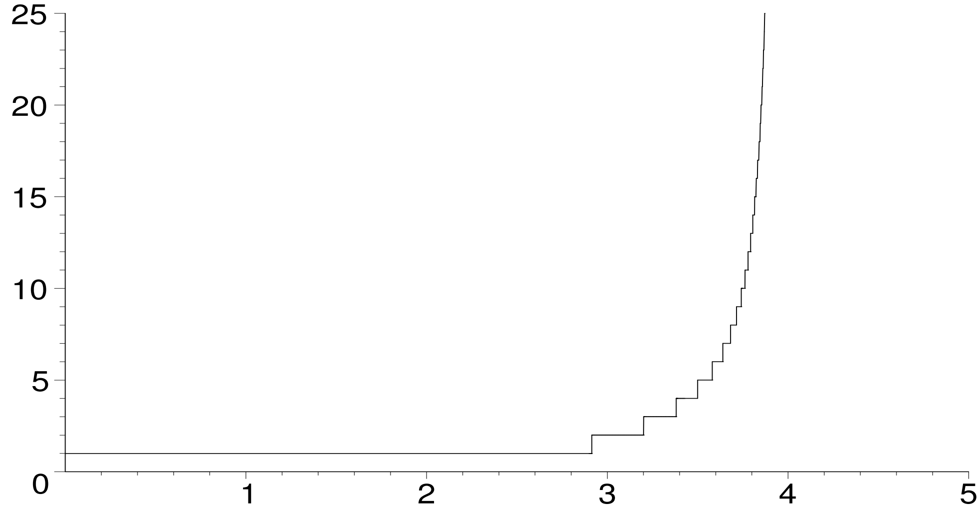

FIGURE 2. The graph of r versus the first unstable mode m. Here $N = 2, R = 5$.

We observe that $m \to +\infty$ as $r \to r_R$. However r_R is precisely the value for which $\hat{A} \to +\infty$ and so our analysis breaks down in such a limit. A question arises naturally: are there ring-like solutions which are stable for all m?

In Section 9 we use formal asymptotics and numerical computations to study the regime $\hat{A}\varepsilon^{\frac{1}{2}} = O(1)$. We make the following conjecture.

Conjecture 2.1 *Let r_R be the root of*

$$(J_1 J_{2,R})'(r_R) = 0. \tag{2.20}$$

Let

$$A_c = 2.694\ G_R(r_R, r_R) \tag{2.21}$$

where G_R is given by (2.10).

Suppose that A is just below A_c. Then there exists a ring-like solution whose radius is r_R.

Suppose that A is just above A_c. Then a ring-like solution of radius $r_0 < r_R$ will expand until its radius reaches r_R. It will then split into two concentric rings which will move away from each other.

Remark. Using the far field expansions $I_0(r) \sim r^{-1/2}e^r, K_0(r) \sim r^{-1/2}e^{-r}$, it is easy to see that

$$r_R \sim R - \frac{1}{2}\ln(2R) \quad \text{as} \quad R \to \infty, \tag{2.22}$$

$$G_R(r_R, r_R) \sim \frac{1}{2} + \frac{1}{2r_R} \quad \text{as} \quad R \to \infty. \tag{2.23}$$

Thus for large R we have $A_c \sim 1.347$, which is precisely the critical threshold for when spike splitting in 1-D occurs (see Section 9 or also [**29**], [**23**]).

The fate of the two resulting rings is unclear. In Section 10 we perform some numerical computations of the full two-dimensional system. They suggest that the inner ring evenually breaks up into spots, while the outer ring can remain stable for a long time. It is an open problem to analyse the properties of multi-ring solutions.

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3 Preliminaries: Some Properties of w and $M_R(r)$

In this section, we consider some properties of the functions $w(y)$ and $M_R(r)$. We first state the following facts for $w(y)$.

Lemma 3.1 *(1) The following identities hold*

$$\int_R (w^{'})^2 = \frac{1}{6}\int_R w^3, \tag{3.1}$$

$$\int_R w^3(y)(\int_{-\infty}^y w^2(z)dz)dy = \int_R w^3(y)(\int_y^\infty w^2(z)dz)dy = \frac{1}{2}\int_R w^3 \int_R w^2. \tag{3.2}$$

(2) The solution to the following problem

$$L_0\phi := \phi^{''} - \phi + 2w^{'}\phi = 0, \quad |\phi| \le C \tag{3.3}$$

for some $C > 0$ *is given by* $\phi = cw^{'}$ *for some constant* c.

(3) The eigenvalues of L_0 *can be arranged as follows:*

$$\lambda_1 = \frac{5}{4}, \quad \lambda_2 = 0, \quad \lambda_3 = -\frac{3}{4}.$$

Proof: (1) can be proved by direct computations. A more general proof can be given as follows: since w satisfies (2.9), we have $(w^{'})^2 = w^2 - \frac{2}{3}w^3$. Combining this with the following identity

$$\int_R ((w^{'})^2 + w^2) = \int_R w^3$$

we deduce (3.1). To prove (3.2), we observe that

$$\int_R w^3(y)(\int_{-\infty}^y w^2(z)dz - \int_y^\infty w^2(z)dz)dy = 0$$

$$\int_R w^3(y)(\int_{-\infty}^y w^2(z)dz + \int_y^\infty w^2(z)dz)dy = \int_R w^3 \int_R w^2.$$

Equation (3.2) then follows.

The statement (2) follows from standard ODE theory.

The statement (3) can be proved by using hypergeometric function. See [**10**].

□

The following lemma characterizes the eigenvalues of Nonlocal Eigenvalue Problem (NLEP)

$$L\phi := \phi^{''} - \phi + 2w\phi - \mu \frac{\int_R w\phi}{\int_R w^2} w^2 = \lambda\phi, \phi \in H^2(R). \tag{3.4}$$

Lemma 3.2 *(1) If $\mu \neq 1$ and*

$$L\phi = 0, \phi \in H^2(R)$$

then $\phi = cw^{'}$ for some constant c.

(2) If $\mu < 1$, then there exists a positive eigenvalue $\lambda_0 > 0$ to (3.4).

(3) If $\mu > 1$, then there exists a constant $C > 0$ such that $Re(\lambda) < -C < 0$, where $\lambda \neq 0$ is an eigenvalue of (3.4).

Proof:

(1) In fact, let $L\phi = 0$. Then we have

$$L_0(\phi - 2\frac{\int_R w\phi}{\int_R w^2} w) = 0$$

where

$$L_0\phi := \phi^{''} - \phi + 2w\phi. \tag{3.5}$$

Since $\phi \in H^2(R)$, by (2) of Lemma 2.1, we have

$$\phi - 2\frac{\int_R w\phi}{\int_R w^2} w = cw^{'} \tag{3.6}$$

for some c. Multiplying (3.6) by w^{m-1} and integrating over R, we obtain that

$$\int_R w^{m-1}\phi = 0$$

which implies that $\phi = cw^{'}$.

(2) and (3) follows from Theorem 1.4 of [**46**].

□

Our next lemma concern again a nonlocal eigenvalue problem in which the coefficient μ depends on $\tau\lambda$. We consider the following nonlocal eigenvalue problem

$$\phi^{''} - \phi + 2w\phi - \chi(\tau\lambda)\frac{\int_R w\phi}{\int_R w^2} w^2 = \lambda\phi \tag{3.7}$$

Lemma 3.3 *Suppose $\chi(z)$ is a continuous function of z. Then*

(1) if $\chi(0) < 1$, there exists a positive eigenvalue $\lambda > 0$ to (3.7).

(2) If $\chi(0) > 1$, and the following condition holds for τ:

$$Re\left[\bar{\lambda}\chi(\tau\lambda) - \lambda\right] + 6|\chi(\tau\lambda) - 1|^2 \geq 0 \tag{3.8}$$

then there exists a positive constant $C > 0$ such that for all nonzero eigenvalue λ of (3.7) we have $Re(\lambda) \leq -C < 0$.

Proof:

Suppose $\chi(0) < 1$. We can solve (3.7) explicitly. We may assume that ϕ is even. We look for a positive eigenvalue λ_0 in $(0, \lambda_1)$ where λ_1 is the first eigenvalue of $L_0\phi = \phi'' - \phi + 2w\phi$. (See Lemma 3.1.) Then (3.7) is equivalent to the following algebraic equation:

$$\rho(\lambda) = \int_R w(L_0 - \lambda)^{-1} w^2 - \frac{\int_R w^2}{\chi(\tau\lambda)} = \left(1 - \frac{1}{\chi(\tau\lambda)}\right) \int_R w^2 + \lambda \int_R w(L_0 - \lambda)^{-1} w. \tag{3.9}$$

Then

$$\rho(0) < 0, \rho(t) \to +\infty \text{ as } t \to \lambda_1,\ t < \lambda_1.$$

Thus $\rho(t)$ has a zero λ_0 in $(0, \lambda_1)$. This proves (1).

(2) is proved in (2.28) of [**42**].

□

Finally we state the following important properties of M_R, defined in (2.8), which will be used in the proof of Theorem 2.1.

Lemma 3.4 *(1) For each fixed $R > 0$, there exists a $r_R > 0$ such that*

$$(J_1 J_{2,R})' > 0, \quad \text{for } r \in (0, r_R), \quad \text{and} \quad (J_1 J_{2,R})'(r_R) = 0 \tag{3.10}$$

and

$$M_R(r_R) > 0. \tag{3.11}$$

(2) For each fixed $\hat{A} < +\infty$, it holds that

$$M_R(r) < 0, \quad \text{for } r \text{small}. \tag{3.12}$$

Proof: (1) Observe that

$$(J_1 J_{2,R})'(R) = J_1'(R) J_2(R) - J_2'(R) J_1(R) < 0,$$

$$(J_1 J_{2,R})'(r) \to +\infty \text{ as } r \to 0.$$

By continuity, (3.10) is thus proved. This then yields that $M_R(r_R) > 0$, recalling

$$M_R(r) = \frac{N-1}{r} + \frac{1-\xi}{\xi} \frac{(J_1 J_{2,R})'}{J_1 J_{2,R}}$$

where ξ satisfies

$$\xi(1-\xi) = \frac{J_1 J_{2,R}}{\hat{A}^2(J_1' J_2 - J_1 J_2')}, \quad 0 < \xi < \frac{1}{2}. \tag{3.13}$$

(2) For r small and $N \geq 3$

$$\frac{J_1 J_{2,R}}{\hat{A}^2(J_1' J_2 - J_1 J_2')} \sim r, \quad 0 < \xi < \frac{1}{2}, \quad \xi \sim r,$$

$$\frac{(J_1 J_{2,R})'}{J_1 J_{2,R}} = -(N-2)\frac{1}{r} + O(1)$$

and hence

$$M_R(r) \sim \frac{N-1}{r} - O(\frac{1}{r^2}) < 0$$

For $N = 2$, we obtain similarly that for r small

$$M_R(r) \sim \frac{N-1}{r} - O(\frac{1}{r^2(\log\frac{1}{r})^2}) < 0.$$

This proves (2) of the lemma.

□

4 A linear Problem

Fixing a point $t > 0$, we set

$$I_\varepsilon := (-\frac{t}{\varepsilon}, \frac{R-t}{\varepsilon}). \tag{4.1}$$

Let $\eta(s)$ be a function such that $\eta(s) = 1$ for $|s| \leq \frac{\delta}{4}$ and $\eta(s) = 0$ for $|s| \geq \frac{\delta}{2}$, where $\delta > 0$ is a fixed small constant. Set

$$w_{\varepsilon,t}(y) = w(y)\eta(1 + \frac{\varepsilon y}{t}). \tag{4.2}$$

Let $\xi_\varepsilon(t)$ be such that

$$1 - \xi_\varepsilon = \frac{1}{\hat{A}^2\xi_\varepsilon} G_R(t;t), \quad 0 < \xi_\varepsilon < \frac{1}{2}. \tag{4.3}$$

We rescale

$$r = t + \varepsilon y, v = \frac{1}{\hat{A}\xi_\varepsilon}\hat{v}. \tag{4.4}$$

Dropping the hat, we see that (1.7) is equivalent to

$$\begin{cases} \varepsilon^2\Delta v - v + \frac{u}{\xi_\varepsilon}v^2 = 0, \text{ in } B_R(0), \\ \Delta u - c_\varepsilon u v^2 + (1-u) = 0, \text{ in } B_R(0), \\ u > 0, v > 0 \text{ in } B_R(0), \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_R(0), \end{cases} \tag{4.5}$$

where

$$c_\varepsilon = (6\varepsilon\hat{A}^2\xi_\varepsilon^2)^{-1}. \tag{4.6}$$

From now on, we shall work with (4.5) instead.

In the sequel, we denote by $T[h]$ the unique solution of the equation

$$\begin{cases} \Delta u + 1 - u - c_\varepsilon h^2 u = 0 & \text{in } B_R(0), \\ u(x) = u(|x|), \quad u^{'}(R) = 0, \end{cases} \tag{4.7}$$

for $h \in L^\infty(B_R(0))$.

The equation (4.7) can be solved by using the Green's function $G_R(r, r_0)$ defined in (2.10) of Theorem 2.1. In fact, the operator T can be written in the following way:

$$1 - T[h](r_0) = c_\varepsilon \int_0^R G_R(r; r_0)(\frac{r}{r_0})^{N-1} h^2(r) T[h](r) dr. \tag{4.8}$$

In this section, we study the operator $T[h]$, where we choose h to be

$$h = \left(w_{\varepsilon,t}(\frac{r-t}{\varepsilon}) + \phi(\frac{r-t}{\varepsilon}) \right)^2, \quad \phi = O(\varepsilon^\sigma), \tag{4.9}$$

for a fixed $0 < \sigma < 1$.

Let

$$T[h](r^{'}) = v_\varepsilon(r^{'}).$$

By definition, we then have

$$1 - T[h](r^{'}) = c_\varepsilon \int_0^R G_R(r; r^{'})(\frac{r}{r^{'}})^{N-1} h^2 v_\varepsilon(r) dr,$$

and hence

$$1 - v_\varepsilon(t) = \frac{1}{\hat{A}^2 \xi_\varepsilon^2} G_R(t; t) v_\varepsilon(t) + O(\varepsilon^\sigma). \tag{4.10}$$

Here we have used the fact that

$$\int_R w^2 = 6. \tag{4.11}$$

From (4.3) and (4.10), we arrive at the following

$$v_\varepsilon(t) = \xi_\varepsilon + O(\varepsilon^\sigma). \tag{4.12}$$

Let $r^{'} = t + \varepsilon y$. Then we have

$$\begin{aligned} 1 - v_\varepsilon(t + \varepsilon y) &= c_\varepsilon \int_0^R G_R(z; t + \varepsilon y) \frac{z^{N-1}}{(t + \varepsilon y)^{N-1}} (w_{\varepsilon,t}(\frac{z-t}{\varepsilon}) + \phi(\frac{z-t}{\varepsilon}))^2 v_\varepsilon(z) dz \\ &= \varepsilon c_\varepsilon \int_{-\frac{t}{\varepsilon}}^{\frac{R-t}{\varepsilon}} G_R(t + \varepsilon z; t + \varepsilon y) \frac{(t + \varepsilon z)^{N-1}}{(t + \varepsilon y)^{N-1}} (w_{\varepsilon,t}(z) + \phi(z))^2 v_\varepsilon(t + \varepsilon z) dz \\ &= E_1 + E_2 \end{aligned}$$

where

$$E_1 = \varepsilon c_\varepsilon \int_{-\frac{t}{\varepsilon}}^{\frac{R-t}{\varepsilon}} G_R(t + \varepsilon z; t + \varepsilon y) \frac{(t + \varepsilon z)^{N-1}}{(t + \varepsilon y)^{N-1}} w_{\varepsilon,t}^2 v_\varepsilon(t + \varepsilon y) dz,$$

$$E_2 = \varepsilon c_\varepsilon \int_{-\frac{t}{\varepsilon}}^{\frac{R-t}{\varepsilon}} G_R(t+\varepsilon z; t+\varepsilon y) \frac{(t+\varepsilon z)^{N-1}}{(t+\varepsilon y)^{N-1}} \left[(w_{\varepsilon,t}(z)+\phi(z))^2 - (w_{\varepsilon,t}(z))^2 \right] v_\varepsilon(t+\varepsilon z) dz.$$

Observe that $G_R(r; r^{'})$ can be expanded as follows: for $\bar{z} < \bar{y}$, we have

$$G_R(t+\bar{z}; t+\bar{y}) = G_R(t;t) \left(1 + (\frac{J_{2,R}^{'}(t)}{J_{2,R}(t)} + \frac{N-1}{t})\bar{y} + \frac{J_1^{'}(t)}{J_1(t)}\bar{z} + O(|\bar{y}|^2 + |\bar{z}|^2) \right). \tag{4.13}$$

For $\bar{y} < \bar{z}$, there is another expansion

$$G_R(t+\bar{z}; t+\bar{y}) = G_R(t;t) \left(1 + (\frac{J_1^{'}(t)}{J_1(t)} + \frac{N-1}{t})\bar{y} + \frac{J_{2,R}^{'}(t)}{J_{2,R}(t)}\bar{z} + O(|\bar{y}|^2 + |\bar{z}|^2) \right). \tag{4.14}$$

Then we have, using (4.12) and (4.13)-(4.14),

$$E_1 = \varepsilon c_\varepsilon \int_{-\frac{t}{\varepsilon}}^{\infty} G_R(t+\varepsilon z; t+\varepsilon y) \frac{(t+\varepsilon z)^{N-1}}{(t+\varepsilon y)^{N-1}} w^2 v_\varepsilon(t+\varepsilon z) dz$$

$$= \varepsilon c_\varepsilon \int_R G_R(t+\varepsilon z; t+\varepsilon y) \left(1 + \frac{\varepsilon(N-1)(z-y)}{t} + O(\varepsilon^2) \right) w^2 v_\varepsilon(t+\varepsilon z) dz$$

$$= \alpha_\varepsilon + \varepsilon\rho(y) + O(\varepsilon^2|y|^2) \tag{4.15}$$

where

$$\alpha_\varepsilon = \varepsilon c_\varepsilon G_R(t;t) \int_R w^2 v_\varepsilon(t+\varepsilon z) = \frac{G_R(t,t)v_\varepsilon(t)}{\hat{A}^2\xi_\varepsilon^2} + O(\varepsilon) = 1 - \xi_\varepsilon + O(\varepsilon^\sigma) \tag{4.16}$$

and $\rho(y)$ is defined by

$$\rho(y) = \frac{\alpha_\varepsilon}{\int_R w^2} \left[\frac{J_{2,R}^{'}(t)}{J_{2,R}(t)} (y \int_{-\infty}^{y} w^2 + \int_y^\infty z w^2(z) dz) + \frac{J_1^{'}(t)}{J_1(t)} (y \int_y^\infty w^m + \int_{-\infty}^{y} z w^2(z) dz) \right]. \tag{4.17}$$

For E_2, we have

$$E_2 = 2\varepsilon c_\varepsilon \int_{-\frac{t}{\varepsilon}}^{\frac{R-t}{\varepsilon}} G_R(t+\varepsilon z; t+\varepsilon y) \frac{(t+\varepsilon z)^{N-1}}{(t+\varepsilon y)^{N-1}} [w\phi] v_\varepsilon(t+\varepsilon z) dz + O(\varepsilon^{2\sigma})$$

$$= \frac{2\alpha_\varepsilon}{\int_{I_\varepsilon} w_{\varepsilon,t}^2} \int_{I_\varepsilon} w_{\varepsilon,t}\phi + O(\varepsilon^{2\sigma}).$$

Summarizing all of these estimates, we have obtained the following lemma:

Lemma 4.1 *For $r^{'} = t + \varepsilon y$, we have*

$$T[(w_{\varepsilon,t}+\phi)^2](t+\varepsilon y) = 1 - \alpha_\varepsilon - \varepsilon\rho(y) - \frac{2\alpha_\varepsilon}{\int_{I_\varepsilon} w^2} \int_{I_\varepsilon} w_{\varepsilon,t}\phi + O(\varepsilon^2|y|^2 + \varepsilon^{2\sigma}). \tag{4.18}$$

5 A Nonlinear Problem: Finite dimensional Reduction

In this section, we perform a Liapunov-Schmidt reduction procedure. Such a reduction method has been introduced and used in many previous studies of spike and layered solutions. See [**1**], [**5**], [**6**], [**7**], [**15**], [**18**], [**47**], [**48**] and the references therein.

For $u, v \in H^1(B_{\frac{R}{\varepsilon}}(0))$, we equip it with the following scalar product:

$$(u, v)_\varepsilon = \int_{I_\varepsilon} (u^{'} v^{'} + uv)(t + \varepsilon y)^{N-1} dy \tag{5.1}$$

(which is equivalent to the norm $\|u\|_{H^1(B_{\frac{R}{\varepsilon}}(0))}$).

Then orthogonality to the function $w^{'}_{\varepsilon,t}$ in that space is equivalent to, setting

$$Z_{\varepsilon,t} = w^{'''}_{\varepsilon,t} + \frac{\varepsilon(N-1)}{t+\varepsilon y} w^{''}_{\varepsilon,t} - w^{'}_{\varepsilon,t} \tag{5.2}$$

to the orthogonality in $L^2(I_\varepsilon)$, equipped with the following scalar product

$$< u, v >_\varepsilon = \int_{I_\varepsilon} (uv)(t + \varepsilon y)^{N-1} dy \tag{5.3}$$

(which is equivalent to the norm $\|u\|_{L^2(\mathbb{R}^N)}$).

To this end, we need to define a norm:

$$\|u\|_* = \|u(y)\|_{L^\infty(I_\varepsilon)}. \tag{5.4}$$

The following Proposition will be proved in Appendix A.

Proposition 5.1 *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, given any $h \in L^\infty(I_\varepsilon)$, there exists a unique pair (ϕ, c) such that the following hold:*

$$\phi^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi^{'} - \phi + 2w_{\varepsilon,t}\phi - 2(1-\xi)\frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi}{\int_{I_\varepsilon} w^2_{\varepsilon,t}} = h + cZ_{\varepsilon,t}, \tag{5.5}$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = 0, \quad \phi^{'}(\frac{R-t}{\varepsilon}) = 0, \quad < \phi, Z_{\varepsilon,t} >_\varepsilon = 0. \tag{5.6}$$

Moreover, we have

$$\|\phi\|_* \le C\|h\|_*. \tag{5.7}$$

In this section, we solve the following system of equations for (ϕ, β):

$$(w_{\varepsilon,t} + \phi)^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}(w_{\varepsilon,t} + \phi)^{'} - (w_{\varepsilon,t} + \phi) \tag{5.8}$$

$$+\frac{1}{\xi_\varepsilon}(w_{\varepsilon,t} + \phi)^2 (T[(w_{\varepsilon,t} + \phi)^2](t + \varepsilon y)) = \beta Z_{\varepsilon,t},$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = 0, \quad \phi^{'}_\varepsilon(\frac{R-t}{\varepsilon}) = 0, \quad \int_{I_\varepsilon} \phi Z_{\varepsilon,t}(t + \varepsilon y)^{N-1} = 0. \tag{5.9}$$

The main result in this section is to show the following proposition:

Proposition 5.2 *For $0 < t < R$ and ε sufficiently small, there exists a unique pair $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$satisfying (5.8)-(5.9). Furthermore, $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$ is continuous in t and we have the following estimate*

$$\|\phi_{\varepsilon,t}\|_* \le \varepsilon^\sigma, \tag{5.10}$$

where $\sigma \in (\frac{1}{2}, 1)$ is a constant.

Proof: We write (5.8) in the following form:

$$L_\varepsilon[\phi] = E_\varepsilon + M_\varepsilon[\phi] + \beta Z_{\varepsilon,t} \tag{5.11}$$

and use contraction mapping theorem. Here

$$E_\varepsilon = -\frac{\varepsilon(N-1)}{t+\varepsilon y} w_{\varepsilon,t}^{'} + \frac{1}{\xi_\varepsilon} w_{\varepsilon,t}^2 (T[w_{\varepsilon,t}^2]) - w_{\varepsilon,t}^2 \tag{5.12}$$

and $M_\varepsilon[\phi]$ is given by

$$M_\varepsilon[\phi] = \frac{1}{\xi_\varepsilon}(w_{\varepsilon,t}+\phi)^2 (T[(w_{\varepsilon,t}+\phi)^2]) - \frac{1}{\xi_\varepsilon} w_{\varepsilon,t}^2 (T[w_{\varepsilon,t}^2]) - 2w_{\varepsilon,t}\phi - 2(1-\xi_\varepsilon)\frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi}{\int_{I_\varepsilon} w_{\varepsilon,t}} w_{\varepsilon,t}^2. \tag{5.13}$$

By Lemma 4.1, it is easy to see that

$$\|E_\varepsilon\|_* \le C\varepsilon^\sigma. \tag{5.14}$$

For M_ε, we note that

$$\begin{aligned} M_\varepsilon[\phi] &= \xi_\varepsilon^{-1}[(w_{\varepsilon,t}+\phi)^2 - w_{\varepsilon,t}^2 - 2w_{\varepsilon,t}\phi](T[(w_{\varepsilon,t}+\phi)^2]) \\ &-\left(w_{\varepsilon,t}^2 (T[w_{\varepsilon,t}^2]) - w_{\varepsilon,t}^2 (T[(w_{\varepsilon,t}+\phi)^2]) - 2\alpha_\varepsilon \frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi}{\int_{I_\varepsilon} w_{\varepsilon,t}^2} w_{\varepsilon,t}^2 \right) \\ &-2w_{\varepsilon,t}\phi\left(1 - \frac{(T[(w_{\varepsilon,t}+\phi)^2])}{\xi_\varepsilon}\right). \end{aligned}$$

The first term in $M_\varepsilon[\phi]$ can be estimated as follows

$$|(w_{\varepsilon,t}+\phi)^2 - w_{\varepsilon,t}^2 - 2w_{\varepsilon,t}\phi|(T[(w_{\varepsilon,t}+\phi)^2])| \le C|\phi|^2.$$

By Lemma 4.1, it follows that second term and the last term in $M_\varepsilon[\phi]$ can be bounded by $O(\varepsilon\|\phi\|_* + \|\phi\|_*^2)$.

Therefore, we have

$$\|M_\varepsilon[\phi]\|_* \le C\left(\varepsilon\|\phi\|_* + \|\phi_\varepsilon\|_*^2\right). \tag{5.15}$$

Set $\mathcal{B} = \{\|\phi\|_* < C\varepsilon^\sigma\}$ where C is large. Fix $\phi \in \mathcal{B}$ and we consider the map $\mathcal{A}_\varepsilon$ to be the unique solution given by Proposition 4.2 with $h = E_\varepsilon + M_\varepsilon[\phi]$. Then by Proposition 4.2, we have

$$\|\mathcal{A}_\varepsilon[\phi]\|_* \le C\|E_\varepsilon + M_\varepsilon[\phi]\|_{**} \le C\varepsilon^\sigma + \varepsilon^{2\sigma} \le C\varepsilon^\sigma, \tag{5.16}$$

and hence $\mathcal{A}_\varepsilon[\phi] \in \mathcal{B}$. Moreover, we also have that

$$\|\mathcal{A}_\varepsilon[\phi_1] - \mathcal{A}_\varepsilon[\phi_2]\|_* \le C\|M_\varepsilon[\phi_1] - M_\varepsilon[\phi_2]\|_* < \|\phi_1 - \phi_2\|_*. \tag{5.17}$$

(5.16) and (5.17) show that the map $\mathcal{A}_\varepsilon$ is a contraction map from $\mathcal{B}$ to $\mathcal{B}$. By the contraction mapping theorem, (5.11) has a unique solution $\phi \in B$, called $\phi_{\varepsilon,t}$.

The continuity of $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$ follows from the uniqueness of $(\phi_{\varepsilon,t}, \beta_\varepsilon(t))$.

□

6 The reduced problem: Proof of Theorem 2.1

In this section we solve the reduced problem and prove our main existence result given by Theorem 2.1.

In particular, we prove that

Proposition 6.1 *For ε sufficiently small, $\beta_\varepsilon(t)$ is continuous in t and we have*

$$\beta_\varepsilon(t) = c_0\varepsilon(M_R(t)) + O(\varepsilon^{2\sigma}), \tag{6.1}$$

for some generic constant $c_0 \neq 0$.

From Proposition 6.1, we can finish the proof of Theorem 2.1.

Proof of Theorem 2.1:

By Lemma 3.4, there exists two numbers $0 < r_1 < r_2 < r_R$ such that

$$M_R(r_1)M_R(r_2) < 0.$$

Since $\varepsilon^{-1}\beta_\varepsilon(t) = c_0 M_R(t) + O(\varepsilon^{2\sigma-1})$, for ε sufficiently small, we also have $\beta_\varepsilon(r_1)\beta_\varepsilon(r_2) < 0$. By the continuity of $\beta_\varepsilon(t)$ and the intermediate mean value theorem, a zero of β_ε is thus guaranteed. □

We now prove Proposition 6.1. To this end, we let $v_{\varepsilon,t} = T[(w_{\varepsilon,t} + \phi_{\varepsilon,t})^2]$. Then $\phi_{\varepsilon,t}$ satisfies

$$\phi_{\varepsilon,t}^{''} - \phi_{\varepsilon,t} + 2w_{\varepsilon,t}\phi_{\varepsilon,t} \tag{6.2}$$

$$= -\frac{\varepsilon(N-1)}{t+\varepsilon y} w_{\varepsilon,t}^{'} + w_{\varepsilon,t}^2 - \xi_\varepsilon^{-1} w_{\varepsilon,t}^2 v_{\varepsilon,t} + N_\varepsilon[\phi_{\varepsilon,t}] + \beta_\varepsilon(t) Z_{\varepsilon,t}$$

where

$$N_\varepsilon[\phi_{\varepsilon,t}] = -\frac{\varepsilon(N-1)}{t+\varepsilon y}\phi_{\varepsilon,t}^{'} + \xi_\varepsilon^{-1}((w_{\varepsilon,t}+\phi_{\varepsilon,t})^2 - w_{\varepsilon,t}^2)v_{\varepsilon,t} - 2w_{\varepsilon,t}^{p-1}\phi_{\varepsilon,t}.$$

Note that by Lemma 4.1

$$v_{\varepsilon,t}(t+\varepsilon y) = 1 - \alpha_\varepsilon - \varepsilon\rho(y) + O(\varepsilon^2|y|^2) - 2\alpha_\varepsilon \frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi_{\varepsilon,t}}{\int_{I_\varepsilon} w_{\varepsilon,t}^2} + O(\varepsilon^{2\sigma}). \tag{6.3}$$

Multiplying the equation for $\phi_{\varepsilon,t}$ by $w_{\varepsilon,t}^{'}$ and integrating over I_ε, we obtain that

$$\beta_\varepsilon(t)\int_{I_\varepsilon} Z_{\varepsilon,t} w_{\varepsilon,t}^{'} = \int_{I_\varepsilon}[\phi_{\varepsilon,t}^{''} - \phi_{\varepsilon,t} + 2w_{\varepsilon,t}\phi_{\varepsilon,t}]w_{\varepsilon,t}^{'} \tag{6.4}$$

$$+\int_{I_\varepsilon}\frac{\varepsilon(N-1)}{t+\varepsilon y}\phi_{\varepsilon,t}^{'} w_{\varepsilon,t}^{'} + \int_{I_\varepsilon}(\xi_\varepsilon^{-1}w_{\varepsilon,t}^2 v_{\varepsilon,t} - w_{\varepsilon,t}^2)w_{\varepsilon,t}^{'} + \int_{I_\varepsilon} N_\varepsilon[\phi_{\varepsilon,t}]w_{\varepsilon,t}^{'}.$$

We first estimate:

$$\int_{I_\varepsilon}[\phi_{\varepsilon,t}^{''} - \phi_{\varepsilon,t} + 2w_{\varepsilon,t}\phi_{\varepsilon,t}]w_{\varepsilon,t}^{'} = O(\varepsilon^{1+\sigma}).$$

Then using (6.3) we obtain

$$\int_{I_\varepsilon} w'_{\varepsilon,t} N_\varepsilon[\phi_{\varepsilon,t}] = -\int_{I_\varepsilon} (\frac{\varepsilon(N-1)}{t+\varepsilon y}\phi'_{\varepsilon,t} w'_{\varepsilon,t}) + \int_{I_\varepsilon} \xi_\varepsilon^{-1}[(w_{\varepsilon,t}+\phi_{\varepsilon,t})^2 - w_{\varepsilon,t}^2 - 2w_{\varepsilon,t}\phi_{\varepsilon,t}]v_{\varepsilon,t}w'_{\varepsilon,t}$$
$$+ 2\int_{I_\varepsilon} w_{\varepsilon,t}\phi_{\varepsilon,t}(\frac{v_{\varepsilon,t}}{\xi_\varepsilon} - 1)w'_{\varepsilon,t}$$
$$= O(\varepsilon^{1+\sigma} + \varepsilon^{2\sigma}).$$

The main term is the following:

$$\int_{I_\varepsilon} w'_{\varepsilon,t} w^2_{\varepsilon,t}(v_{\varepsilon,t} - \xi_\varepsilon) = \varepsilon(\int_R w' w^2 \rho(y) + O(\varepsilon^{2\sigma})). \tag{6.5}$$

By using (3.2) of Lemma 3.1 we calculate

$$\int_R w^2 \int_R w' w^2 \rho(y) = \alpha_\varepsilon \frac{J'_{2,R}(t)}{J_{2,R}(t)} \int_R w' w^2 (y\int_{-\infty}^y w^2 + \int_{-\infty}^y zw^2(z))$$
$$+ \frac{J'_1(t)}{J_1(t)} \int_R w' w^2 (y\int_y^\infty w^2 + \int_y^\infty zw^2(z))$$
$$= -\frac{\alpha_\varepsilon}{3}\frac{J'_{2,R}(t)}{J_{2,R}(t)} \int_R w^3(y)\int_{-\infty}^y w^2 - \frac{1}{3}\frac{J'_1(t)}{J_1(t)}\int_R w^3 \int_y^\infty w^2$$
$$= -\frac{\alpha_\varepsilon}{6}(\frac{J'_{2,R}(t)}{J_{2,R}(t)} + \frac{J'_1(t)}{J_1(t)}) \int_R w^3(y) \int_R w^2. \tag{6.6}$$

Combining all of these expressions, we obtain

$$\beta_\varepsilon(t)\int_R (w(w')^2) = \varepsilon\frac{N-1}{t}\int_R (w')^2 + \varepsilon\frac{\alpha_\varepsilon}{\int_R w^2\xi_\varepsilon}\int_R w^3(\frac{J'_{2,R}(t)}{J_{2,R}(t)} + \frac{J'_1(t)}{J_1(t)}) + O(\varepsilon^{2\sigma})$$
$$= \varepsilon\int_R (w')^2\left[M_R(r) + O(\varepsilon^{2\sigma-1})\right]$$

using (3.1) of Lemma 3.1.

□

7 Proofs of Theorem 2.2 and Theorem 2.3

In this section, we use Lemmas 3.2 and 3.3, and apply a compactness argument of Dancer [**9**] to prove Theorems 2.2 and 2.3.

We consider large eigenvalues of the following problem

$$\begin{cases} \varepsilon^2\Delta\phi_\varepsilon - \frac{\varepsilon^2 m^2}{r^2}\phi_\varepsilon - \phi_\varepsilon + 2\hat{A}v_{\varepsilon,R}u_{\varepsilon,R}\phi_\varepsilon + \hat{A}^2 v^2_{\varepsilon,R}\psi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon, \\ \Delta\psi_\varepsilon - \frac{m^2}{r^2}\psi_\varepsilon - \psi_\varepsilon - 2\varepsilon^{-1}v_{\varepsilon,R}u_{\varepsilon,R}\phi_\varepsilon - \varepsilon^{-1}v^2_{\varepsilon,R}\psi_\varepsilon = \tau\lambda_\varepsilon\psi_\varepsilon, \\ \phi_\varepsilon = \phi_\varepsilon(r), \psi_\varepsilon = \psi_\varepsilon(r), \\ \phi'_\varepsilon(R) = \psi'_\varepsilon(R) = 0. \end{cases} \tag{7.1}$$

Namely we assume that $\lambda_\varepsilon \to \lambda_0 \neq 0$ where $\lambda_\varepsilon \in \mathcal{C}$-the set of complex numbers.

We first derive the limiting eigenvalue problem. We may assume that $Re(\lambda_0) \geq 0$ as otherwise we have stability. To this end, let $G_{m,R,\beta}(r;r_0)$ be the Green's function satisfying

$$G^{''} + \frac{N-1}{r}G^{'} - (1+\beta)G - \frac{m^2}{r}G + \delta_{r_0} = 0, \quad \text{for} \quad 0 < r < R, \quad G_R^{'}(R;r_0) = 0, \tag{7.2}$$

which exists if $Re(1+\beta) > 0$.

Assume that

$$\phi_\varepsilon(r_\varepsilon + \varepsilon y) \to \phi_0(y).$$

Then we have

$$\psi_\varepsilon(r_\varepsilon) = \int_0^R G_{m,R,\tau\lambda_\varepsilon}(r;r_\varepsilon)(-2v_{\varepsilon,R}u_{\varepsilon,R} - v_{\varepsilon,R}^2\psi_\varepsilon)\frac{r^{N-1}}{(r_\varepsilon)^{N-1}}dr$$

$$= G_{m,R,\tau\lambda_\varepsilon}(r_\varepsilon;r_\varepsilon)\bigg(- 2\hat{A}^{-1}\int_R w\phi_0 - \hat{A}^{-2}\xi^{-2}\int_R w^2\psi_\varepsilon(r_\varepsilon) + o(1)\bigg).$$

Hence we obtain

$$\psi_\varepsilon(r_\varepsilon) = G_{m,R,\tau\lambda_\varepsilon}(r_\varepsilon;r_\varepsilon)\bigg(- 2\hat{A}^{-1}\int_R w\phi_0 + o(1)\bigg)\bigg(1 + G_{m,R,\tau\lambda_\varepsilon}\hat{A}^{-2}\xi^{-2}\int_R w^2\bigg). \tag{7.3}$$

Substituting (7.3) into the equation for ϕ_ε, we have that ϕ_0 satisfies

$$\phi_0^{''} - \phi_0 + 2w\phi_0 - \mu(\tau\lambda_0)\frac{\int_R w\phi_0}{\int_R w^2} = \lambda_0\phi_0, \tag{7.4}$$

where $\mu(\tau\lambda_0)$ satisfies

$$\mu(\tau\lambda_0) = \frac{(1-\xi)G_{m,R,\tau\lambda_0}(r_0;r_0)}{\xi G_0(r_0,;r_0) + (1-\xi)G_{m,R,\tau\lambda_0}(r_0;r_0)}. \tag{7.5}$$

Recalling Lemma 3.3, we see that if $\mu(0) < 1$, then there exists a positive eigenvalue $\lambda_0 > 0$ to (7.4) for all $\tau > 0$. Note that $\mu(0) < 1$ is equivalent to

$$\frac{G_{m,R}(r_0;r_0)}{G_{0,R}(R_0;r_0)} > \frac{1-\xi}{\xi}. \tag{7.6}$$

Observe that

$$G_{m,R}(r_0;r_0) = \frac{I_m(r_0)K_m(r_0) - \frac{K_m^{'}(R)}{I_m^{'}(R)}I_m^2(r_0)}{I_m^{'}(r_0)K_m(r_0) - K_m^{'}(r_0)I_m(r_0)}.$$

By the Wronskian property of Bessel functions (see [**4**]) we have

$$I_m^{'}K_m - K_m^{'}I_m = \frac{1}{r}.$$

Using this relation we derive,

$$G_{m,R}(r_0;r_0) = r_0\left(I_m(r_0)K_m(r_0) - \frac{K_m^{'}(R)}{I_m^{'}(R)}I_m^2(r_0)\right), \tag{7.7}$$

$$G_{0,R}(r_0;r_0) = r_0\left(I_0(r_0)K_0(r_0) - \frac{K_0^{'}(R)}{I_0^{'}(R)}I_0^2(r_0)\right). \tag{7.8}$$

Substituting (7.7) and (7.8) into (7.6), we see that (7.6) is equivalent to

$$\rho_{m,R}(r_0) < \frac{\xi}{1-\xi}, \tag{7.9}$$

where $\rho_{m,R}$ is defined at (2.15).

To make the above arguments rigorous, we apply arguments of Dancer [**9**]. We write (7.1) as follows:

$$\phi_\varepsilon = \mathcal{R}_\varepsilon \left(2\hat{A} v_{\varepsilon,R} u_{\varepsilon,R} \phi_\varepsilon + \hat{A} v_{\varepsilon,R}^2 T_\varepsilon[\phi_\varepsilon] \right), \tag{7.10}$$

where $T_\varepsilon[\phi_\varepsilon] = \psi_\varepsilon$ and $\mathcal{R}_\varepsilon = (-\varepsilon^2\Delta + (1+\lambda_\varepsilon) - \frac{\varepsilon^2 m^2}{r^2})^{-1}$. We look for solutions of (7.10) with $\lambda_\varepsilon = \lambda_0 + o(1)$. Since $\lambda_0 > 0$, we may assume that $\lambda_\varepsilon > 0$. The operator $\mathcal{R}_\varepsilon$ is certainly compact in the class of radially symmetric functions. By using Dancer's argument, for ε sufficiently small, (7.10) admits a solution $(\phi_\varepsilon, \lambda_\varepsilon)$ where $\lambda_\varepsilon = \lambda_0 + o(1) > 0$.

If $\rho_{m,R}(r_0) < \frac{\xi}{1-\xi}$, then $\mu(0) > 1$. By taking τ small and applying Lemma 3.3, we have proved Theorem 2.2.

Next we prove Theorem 2.3. Note that $\rho_{m,R} = O(\frac{1}{m})$ for $m \gg 1$. So if $1 \ll m \ll \frac{1}{\varepsilon}$, then the instability criteria

$$\rho_{m,R}(r) < \frac{\xi}{1-\xi}$$

is satisfied. The previous arguments in Theorem 2.2 can be applied here.

In the case of $m = \frac{\hat{m}}{\varepsilon} > \frac{c_0}{\varepsilon}$ for some fixed $c_0 > 0$, the equation for ψ_ε becomes

$$\varepsilon^2 \Delta\psi_\varepsilon - \frac{\hat{m}^2}{r_0^2}\psi_\varepsilon - \varepsilon^2\psi_\varepsilon - 2\varepsilon v_{\varepsilon,R} u_{\varepsilon,R}\phi_\varepsilon - \varepsilon v_{\varepsilon,R}^2 \psi_\varepsilon = \tau\varepsilon^2\lambda_\varepsilon\psi_\varepsilon.$$

By the scaling $r = r_\varepsilon + \varepsilon y$, we see that $\psi_\varepsilon(r_\varepsilon + \varepsilon y) \to \psi_0(y)$ which satisfies

$$\Delta\psi_0 - \frac{\hat{m}^2}{r_0^2}\psi_0 = 0,$$

and hence $\psi_0 \equiv 0$.

On the other hand, the equation for $\phi_\varepsilon(r_\varepsilon + \varepsilon y) = \hat{\phi}_\varepsilon(y)$ becomes

$$\hat{\phi}_\varepsilon'' + \frac{\varepsilon}{r_\varepsilon + \varepsilon y}\hat{\phi}_\varepsilon' - \frac{\hat{m}^2}{(r_\varepsilon + \varepsilon y)^2}\hat{\phi}_\varepsilon - \hat{\phi}_\varepsilon + \hat{A} v_{\varepsilon,R} u_{\varepsilon,R}\hat{\phi}_\varepsilon + \hat{A} v_{\varepsilon,R}^2 \psi_\varepsilon = \lambda_\varepsilon \hat{\phi}_\varepsilon,$$

and thus as $\varepsilon \to 0$, $\hat{\phi}_\varepsilon \to \phi_0$ which satisfies

$$\phi_0'' - \phi_0 + 2w\phi_0 = (\lambda_0 + \frac{\hat{m}^2}{r_0^2})\phi_0. \tag{7.11}$$

Since the operator $L_0\phi = \phi'' - \phi + 2w\phi$ has only one positive eigenvalue $\frac{5}{4}$, we then have

$$\lambda_0 + \frac{\hat{m}^2}{r_0^2} \le \frac{5}{4}, \tag{7.12}$$

which proves Theorem 2.3 if $\hat{m} > \frac{\sqrt{5}}{2} r_0$. If $\hat{m} < \frac{\sqrt{5}}{2} r_0$, then (7.11) has a positive eigenvalue $\lambda_0 = \frac{5}{4} - \frac{\hat{m}^2}{r_0^2} > 0$ and hence (7.1) is unstable in this case.

This finishes the proof of Theorem 2.3.

□

8 Generalization to other radially symmetric domains

Theorem 2.1 and Theorem 2.2 can also be generalized to $\mathbb{R}^N$, to the case of an annulus or to the exterior of a ball. Namely, we consider the following problem

$$\begin{cases} \varepsilon^2 \Delta v - v + \hat{A} v^2 u = 0 & \text{in } \Omega, \\ \Delta u + 1 - u - v^2 u = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{8.1}$$

where $\Omega = \mathbb{R}^N$, or $\Omega = B_{R_2}(0) \backslash B_{R_1}(0)$ or $\Omega = \mathbb{R}^N \backslash B_R(0)$.

We consider the general annulus case first. The $\mathbb{R}^N$ case can be considered as a special case of annulus with $R_1 = 0, R_2 = +\infty$ and the exterior of a ball can be considered as annulus with $R_1 = R$ and $R_2 = +\infty$. Let J_1 and J_2 be as defined in (2.1) and (2.2). We then define two new functions J_{1,R_1} and J_{2,R_2}

$$J_{1,R_1}(r) = J_1(r) - \frac{J_1'(R_1)}{J_2'(R_1)} J_2(r), \quad J_{2,R_2}(r) = J_2(r) - \frac{J_2'(R_2)}{J_1'(R_1)} J_1(r) \tag{8.2}$$

and a new Green's function $G_{R_1,R_2}(r; r_0)$

$$G''_{R_1,R_2} + \frac{N-1}{r} G'_{R_1,R_2} - G_{R_1,R_2} + \delta_{r_\varepsilon} = 0; \quad G'_{R_1,R_2}(R_1; r_0) = 0, \quad G'_{R_1,R_2}(R_2; r_0) = 0. \tag{8.3}$$

Similar to (2.6), we have

$$G_{R_1,R_2}(r; r_0) = \frac{1}{\mathcal{W}} \begin{cases} J_{2,R_2}(r_0) J_{1,R_1}(r), & \text{for } R_1 < r < r_0, \\ J_{1,R_1}(r_0) J_{2,R_2}(r), & \text{for } r_0 < r < R_2, \end{cases} \tag{8.4}$$

where

$$\mathcal{W} = J'_{1,R_1}(r_0) J_{2,R_2}(r_0) - J_{1,R_1}(r_0) J'_{2,R_2}(r_0).$$

Define

$$M_{R_1,R_2}(r) := \frac{(N-1)(p-1)}{r} + \frac{1-\xi}{\xi} \frac{(J_{1,R_1} J_{2,R_2})'(r)}{J_{1,R_1}(r) J_{2,R_2}(r)}, \tag{8.5}$$

where ξ satisfies

$$\xi(1-\xi) = \frac{G_{R_1,R_2}(r; r)}{\hat{A}^2}. \tag{8.6}$$

Note that

$$J_{1,0} = J_1, \quad J_{2,+\infty} = J_2, \quad G_{0,R}(r; r_0) = G_R(r; r_0), \quad M_{0,R}(r) = M_R(r). \tag{8.7}$$

Theorem 8.1 *Assume that there exists two points $R_1 < r_1 < r_2 < R_2$ such that*

$$M_{R_1,R_2}(r_1) M_{R_1,R_2}(r_2) < 0. \tag{8.8}$$

Then for ε sufficiently small, problem (8.1) has a radially symmetric solution $(v_{\varepsilon,R_1,R_2}, u_{\varepsilon,R_1,R_2})$ with the following properties:

(1) $v_{\varepsilon,R_1,R_2}, u_{\varepsilon,R_1,R_2}$ are radially symmetric,

(2) $v_{\varepsilon,R_1,R_2}(r) = (1+o(1)) \xi_\varepsilon w(\frac{r-r_\varepsilon}{\varepsilon})$,

(3) $v_{\varepsilon,R_1,R_2}(r) = 1-(1+o(1))\xi_\varepsilon(1+o(1))(G_{R_1,R_2}(r_\varepsilon;r_\varepsilon))^{-1}G_{R_1,R_2}(r;r_\varepsilon)$*, where* $G_{R_1,R_2}(r;r_\varepsilon)$ *satisfies (8.3) and* ξ_ε *is defined by the following relation:*

$$1-\xi_\varepsilon = \frac{G_{R_1,R_2}(r_\varepsilon;r_\varepsilon)}{\hat{A}^2\xi_\varepsilon}, \quad 0<\xi_\varepsilon<\frac{1}{2} \tag{8.9}$$

and $r_\varepsilon \to r_0 \neq 0$ *where* $M_{R_1,R_2}(r_0)=0$*.*

In general, it is difficult to study the function $M_{R_1,R_2}(r)$. A graph of $\hat{A}^2$ and r for $(R_1,R_2)=(3,5)$ is given in Figure 3. The stability of $(v_{\varepsilon,R_1,R_2}, u_{\varepsilon,R_1,R_2})$ can also be studied. Figure 4 shows the relation between r and the minimal m for $R_1=3,\ R_2=5$ by similar methods as before.

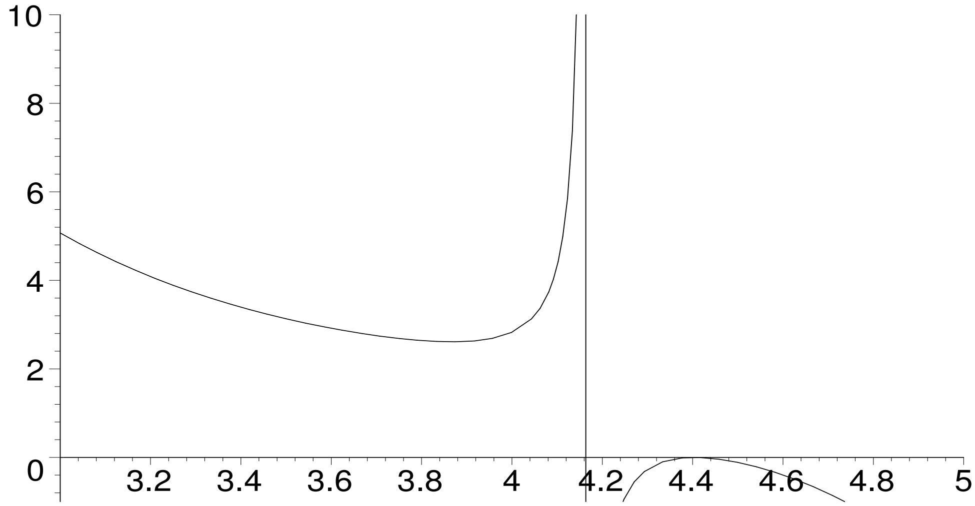


FIGURE 3. A graph of r versus $\hat{A}^2$ for $(R_1,R_2)=(3,5)$.

In the case of $\mathbb{R}^N$, we have a more precise result due to the following lemma.

Lemma 8.1 *There exists* $\hat{A}_c \geq \sqrt{2}$ *such that the function* $M_{0,+\infty}(r)=0$ *has a solution if and only if* $0<\hat{A}<\hat{A}_c$*.*

Remark: We conjecture that $\hat{A}_c = 1.460$ if $N=2$ and $\hat{A}_c=\sqrt{2}$ if $N\geq 3$.

Proof: Observe that $M_{0,+\infty}(r)=0$ if and only if the following holds

$$\hat{A}^2 = -G_{0,+\infty}\frac{(1-\gamma)^2}{\gamma}, \quad \frac{1}{\xi^2-\xi} = \frac{1}{\gamma}-2+\gamma \tag{8.10}$$

where $G_{0,+\infty}=G_{0,+\infty}(r;r)$ is defined in (2.5) and

$$\gamma(r) = \frac{1}{N-1}\frac{r(J_1(r)J_2(r))'}{J_1(r)J_2(r)}. \tag{8.11}$$

First, we show that $\hat{A}^2$, as given by (8.10), is positive for all r, i.e. $\gamma<0$ for all $r>0$. This is equivalent to

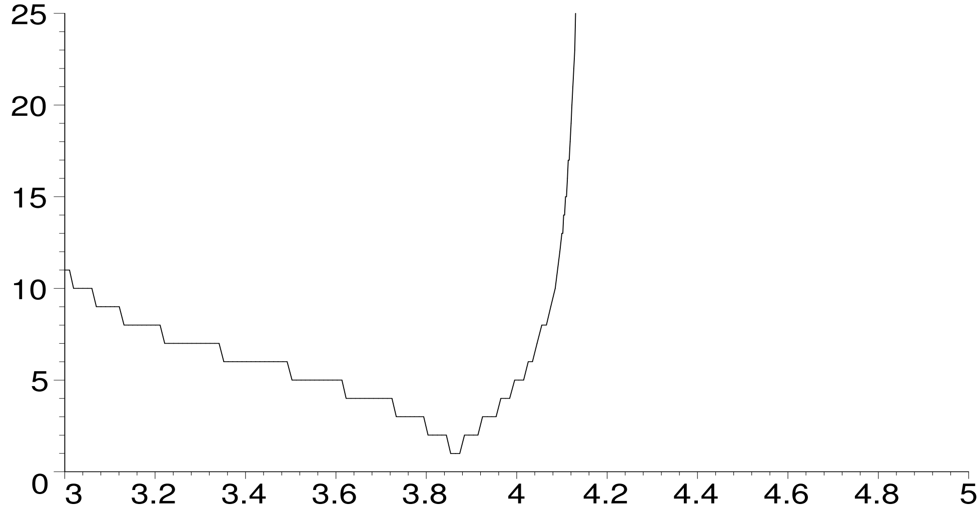


FIGURE 4. A graph r versus minimal m for $(R_1, R_2) = (3, 5)$.

showing that $u = r(J_1 J_2)'$ is always negative. After some algebra, we obtain:

$$u''(r) + \frac{N-1}{r} u'(r) - 4u = 2N J_1 J_2. \tag{8.12}$$

Note that $J_1 J_2 > 0$, $u(r) \sim -(N-1)C\frac{1}{r^{N-1}}$ as $r \to \infty$, and

$$u \sim \begin{cases} -C, & N = 2 \\ -(N-2)Cr^{2-N}, & N > 2 \end{cases} \quad \text{as} \quad r \to 0.$$

In the expression above, C is some positive constant that may change from line to line. Thus u is negative on the boundary of an annulus $\{x : \varepsilon < |x| < R\}$, for any R large enough and for any ε small enough. It then follows from the positivity of $J_1 J_2$ and the comparison principle that u is negative everywhere on that annulus. Since ε and R are arbitrary, $u(r) < 0$ for all $r > 0$.

It remains to show that

$$0 < \xi < 1/2 \quad \text{for all} \quad r. \tag{8.13}$$

But this is immediate from (8.10) since $\gamma < 0$ implies that $\frac{1}{\gamma} - 2 + \gamma \le -4$ which is equivalent to (8.13)

Finally, note that $\hat{A}^2 \to 2$ as $r \to \infty$, $\hat{A}^2 \to 0$ as $r \to \infty$, which proves the existence of $\hat{A}_c \ge \sqrt{2}$. □

Remark. It is easy to see that $\xi = \frac{1}{2}$ at a point r_c such that $\gamma(r_c) = -1$. The latter implies that $G'_{0,+\infty}(r_c; , r_c) = 0$. It then follows from (8.10) that $\frac{d}{dr}\hat{A}^2|_{r=r_c} = 0$. In two dimensions, numerical simulation show that $r_c = 1.075$ is indeed a global maximum $\hat{A}_c$ of $\hat{A}$ with $\hat{A}_c^2 = 4G_{0,+\infty}(1.075; 1.075) = 2.133$. However in three or higher dimensions, numerics indicate that no such r_c exists, and the maximum $\hat{A}_c = \sqrt{2}$ is achieved only at infinity. See Figure 5. That the point r_c exists in two dimensions, can be seen as follows: Simple properties of J_1, J_2 (see for instance [**4**]) yield: $\gamma(0) = -1 + \frac{1}{N-1}$ and $\gamma(r) = -1 + \frac{N-3}{4r^2} + O(\frac{1}{r^3})$. for r large. Hence $\gamma > -1$ for r near 0 and, when $N = 2$, $\gamma < -1$ for large r. This proves the existence of $\gamma(r_c) = -1$ in two dimensions. For $N \geq 3$, we have $\gamma > -1$ near $r = \infty$ and so this argument no longer applies.

Combining Lemma (8.1) with Theorem (8.1) , we obtain the following existence of ground-state solution

Theorem 8.2 *There exists* $\hat{A}_c \geq \sqrt{2}$ *such that for* ε *sufficiently small and for* $0 < \hat{A} < \hat{A}_c$*, the problem (8.1) with* $\Omega = \mathbb{R}^N$ *has a radially symmetric solution* $(v_{\varepsilon,0,+\infty}, u_{\varepsilon,0,+\infty})$ *with the following properties:*

(1) $v_{\varepsilon,0,+\infty}, u_{\varepsilon,0,+\infty}$ *are radially symmetric,*

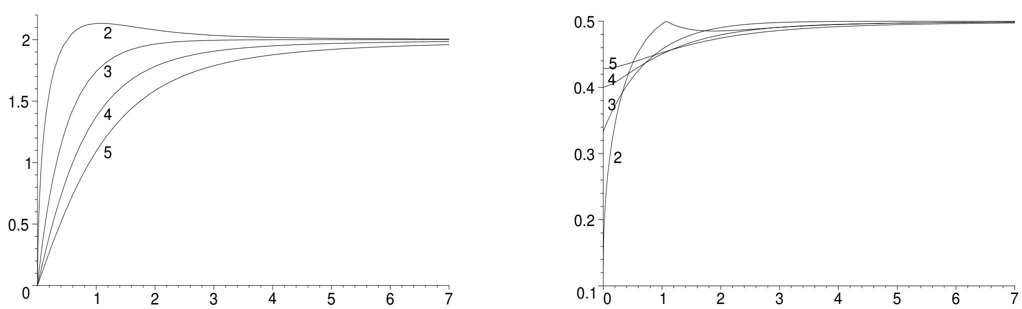
(2) $v_{\varepsilon,0,+\infty}(r) = (1 + o(1))\xi_\varepsilon w(\frac{r-r_\varepsilon}{\varepsilon})$*,*

(3) $v_{\varepsilon,0,+\infty}(r) = 1-(1+o(1))\xi_\varepsilon(1+o(1))(G_{0,+\infty}(r_\varepsilon; r_\varepsilon))^{-1}G_{0,+\infty}(r; r_\varepsilon)$*, where* $G_{0,+\infty}(r; r_\varepsilon)$ *satisfies (8.3) and* ξ_ε *is defined by*

$$1 - \xi_\varepsilon = \frac{G_{0,+\infty}(r_\varepsilon; r_\varepsilon)}{\hat{A}^2 \xi_\varepsilon}, \quad 0 < \xi_\varepsilon < \frac{1}{2}. \tag{8.14}$$

In addition $r_\varepsilon \to r_0 \neq 0$ *as* $\varepsilon \to 0$ *where* $M_{0,+\infty}(r_0) = 0$*. Such a solution dissapears if* $\hat{A} > \hat{A}_c$*.*

In Figure 5a we plot the graph of r versus $\hat{A}^2$ for dimension $N = 2, 3, 4, 5$. It is clear from the graph that $\hat{A}$ is bounded. Note also that in the case of two dimensions only, $\hat{A}$ has a maximum at $r = 1.07$ with $\xi = \frac{1}{2}$ at that point.



(a) r versus $\hat{A}^2$. (b) r versus ξ.

FIGURE 5. (a) The graph of r versus $\hat{A}^2$ with $R = \infty$ and for $N = 2, 3, 4, 5$, as indicated. Note that for $N = 2$, $\hat{A}$ attains a maximum at $r = 1.07$. For $N > 2$, the maximum is attained at infinity. (b) The graph of r versus ξ with $R = \infty$ and N as indicated.

The stability properties for ring solutions in $\mathbb{R}^N$ is dramatically different from the ball case. Our numerical computation shows that $(v_{\varepsilon,0,+\infty}, u_{\varepsilon,0+\infty})$ is always unstable with respect to large eigenvalues for all $1 \leq m \ll \frac{1}{\varepsilon}$.

9 Self-replicating Rings Region: $A = O(1)$.

In this section, we use matched asymptotic analysis to study the case when $A = O(1)$.

Consider again the stability criteria given by (2.17):

$$\rho_{m,R}(r) > \frac{\xi}{1-\xi}.$$

Note that $\rho_{m,R} = O(1/m)$ for large values of m, for a fixed radius r. (See [**4**].) But in order for the ring to be stable, this stability criteria must hold for all m in the range $0 \le m \ll \frac{1}{\varepsilon}$. It follows that a stable ring can only occur if $\xi \le O(\varepsilon)$. However, from (2.11) and (1.6), this corresponds to the regime $\hat{A} = O(\varepsilon^{-\frac{1}{2}}), A = O(1)$, where our analysis breaks down. It follows that a ring is always unstable for the regime $A = O(\varepsilon^{\frac{1}{2}})$. A natural question then, is whether it is possible for a ring to become stable when $A = O(1)$.

When $A = O(1)$, the corresponding one-dimensional problem for the radial profile of the ring becomes coupled as we show below. Moreover, it was shown in [**29**] and [**23**] that this one-dimensional problem exhibits spike splitting, where a spike may split into two spikes, if its distance from the boundary and/or adjacent spikes exceeds a certain threshold. In two dimensions, this corresponds to a ring splitting into two rings. This phenomenon is illustrated below in Figure 11. The mechanism here is essentially the same as the one-dimensional spike-splitting, which we describe here briefly using matched asymptotics (see also [**29**] and [**23**]).

We assume that the ring profile has the shape

$$v(r) = \frac{1}{\varepsilon} W(y), \quad u(r) = \varepsilon \frac{U(y)}{A}, \quad y = \frac{r - r_0}{\varepsilon},$$

where both W, U are of order 1. We then obtain:

$$\begin{aligned} W'' + \varepsilon \frac{W'}{r_0 + \varepsilon y} - W + W^2 U = 0, \\ \frac{1}{\varepsilon^2} U'' + \frac{1}{\varepsilon} \frac{U'}{r_0 + \varepsilon y} - U + \frac{A}{\varepsilon} - \frac{W^2 U}{\varepsilon^2} = 0. \end{aligned}$$

Discarding lower-order terms, the problem for the profile of the ring becomes:

$$W'' - W + W^2 U = 0, \tag{9.1}$$

$$U'' - W^2 U = 0. \tag{9.2}$$

Outside the core region of the ring, we have:

$$\begin{aligned} u(r_0) &= 1 - \int_0^\infty \frac{r}{r_0} G_R(r, r_0) v^2(r) u(r) dr \qquad (9.3) \\ &= 1 - \frac{1}{A} \int_{-\frac{r_0}{\varepsilon}}^\infty \frac{r}{r_0} G_R(r + \varepsilon y, r_0) W^2(y) U(y) dy, \\ \varepsilon \frac{U(0)}{A} &\sim 1 - \frac{G_R(r_0, r_0)}{A} \int_{-\infty}^\infty W^2 U, \end{aligned}$$

where G_R is the radial Green's function on the disk of radius R, given by (2.10). Since $U(0)$ is of order 1, we obtain:

$$\int_{-\infty}^\infty W^2 U = \frac{A}{G_R(r_0, r_0)}.$$

From (9.2), this yields:

$$U'(\infty) - U'(-\infty) = \frac{A}{G_R(r_0, r_0)}.$$

Normally, U will not be symmetric. However on a disk of radius R, it is symmetric for the special case when

$r_0 = r_R$, i.e. when following condition holds:

$$(J_1(r_0)J_{2,R}(r_0))' = 0, \tag{9.4}$$

as we now show.

From (9.3) we obtain:

$$u'(r_0^\pm) \sim -\tfrac{1}{A}\tfrac{d}{dr_0^\pm}\left(\tfrac{1}{r_0^\pm}G_R(r,r_0^\pm)\right)\Big|_{r=r_0} r_0\int_{-\infty}^{\infty} W^2(y)U(y)dy.$$

But the matching condition of the outer solution u and the inner solution U is:

$$U'(\pm\infty) = Au'(r_0^\pm).$$

Thus we obtain:

$$\begin{aligned}
U'(\infty) &= -\frac{1}{J_1'(r_0)J_2(r_0) - J_1(r_0)J_2'(r_0)} r_0 J_1'(r_0)J_{2,R}(r_0)\int_{-\infty}^{\infty} W^2(y)U(y)dy \\
&= -\frac{1}{J_1'(r_0)J_2(r_0) - J_1(r_0)J_2'(r_0)} r_0 J_1'(r_0)J_{2,R}(r_0)\frac{A}{G_R(r_0,r_0)} \\
&= A\frac{J_1'(r_0)J_{2,R}(r_0)}{J_1(r_0)J_{2,R}(r_0)}, \\
U'(-\infty) &= A\frac{J_1(r_0)J_{2,R}'(r_0)}{J_1(r_0)J_{2,R}(r_0)}.
\end{aligned}$$

For U to be symmetric, we must have $U'(\infty) + U'(-\infty) = 0$, which is exactly the condition (9.4).

Note that r_0 given by (9.4) corresponds to the limiting case $\hat{A} \to \infty$ (see (2.13) and the remark following it), which agrees with the regime $A = O(1)$. Thus we conjecture that r_0 in the case $A = O(1)$ will satisfy (9.4).

Assuming U to be symmetric, we thus obtain the following boundary value problem for U, W:

$$U'(0) = 0, \qquad U'(\infty) = \frac{A}{2G_R(r_R, r_R)}.$$

In [**23**] and [**29**] it was shown that the solution to the core problem exists only if $U'(\infty)$ is small enough. To show this, we plot the graph of $U'(\infty)$ versus $\gamma = U(0)W(0)$ below. Numerically, it has a fold point at $U'(\infty) = 1.347,\ \gamma = 1.02$. Thus if we choose $U'(\infty)$ just above 1.347, the ring ceases to exist. Numerically, this corresponds the ring splitting into two rings. The mechanism responsible for this is similar to the mechanism proposed by [**34**] for the case of $\frac{D_v}{D_u} = O(1)$ diffusivity ratio, and is described below. We let

$$\begin{aligned}
v &= \frac{1}{\varepsilon}\left(W + e^{im\theta}e^{t\lambda}\Phi(y)\right), \\
u &= \frac{\varepsilon}{A}\left(U + e^{im\theta}e^{t\lambda}\Psi(y)\right),
\end{aligned}$$

to obtain the following leading-order eigenvalue problem:

$$\begin{aligned}
&\Phi'' - \Phi + 2WV\Phi + W^2\Psi = \lambda\Phi \\
&\Psi'' - 2WV\Phi - W^2\Psi = \tau\lambda\Psi \\
&\Psi'(\pm\infty) = 0 = \Phi(\pm\infty).
\end{aligned} \tag{9.5}$$

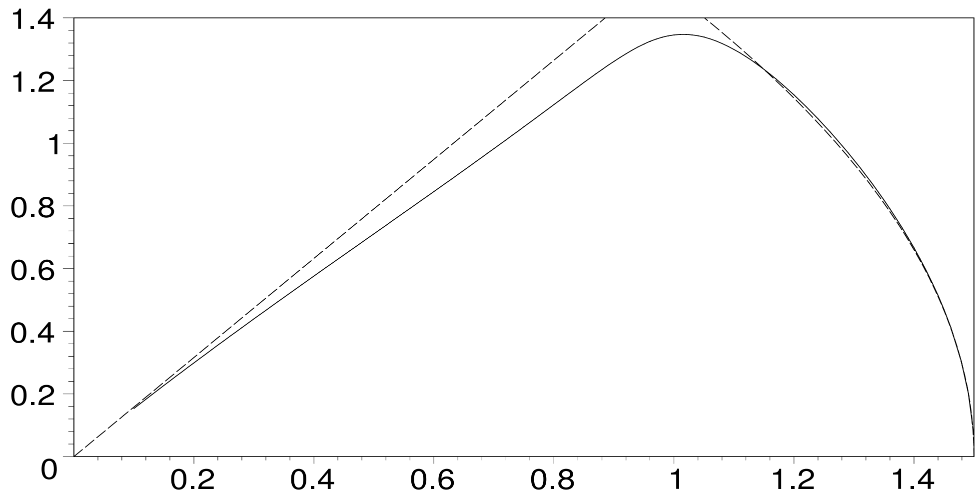


FIGURE 6. The graph of $\gamma = W(0)U(0)$ versus $U'(\infty)$. The fold point occurs at $\gamma = 1.02$, $U'(\infty) = 1.347$. The dashed curves represent an asymptotic approximations, derived in [**23**].

Note that the leading order problem is independent of the mode m. It follows that all modes m with $m \ll \frac{1}{\varepsilon}$ are stable, provided that the zero mode is stable.

At the fold point, we have $\frac{dU'(\infty)}{d\gamma} = 0$ and therefore we see that

$$\lambda = 0, \quad \Phi = \frac{dU}{d\gamma}, \quad \Psi = \frac{dW}{d\gamma},$$

satisfies (9.5). Solving this system numerically, we obtain a solution for Φ that has a dent, as shown in Figure 7. This is the eigenfunction that is responsible for ring splitting.

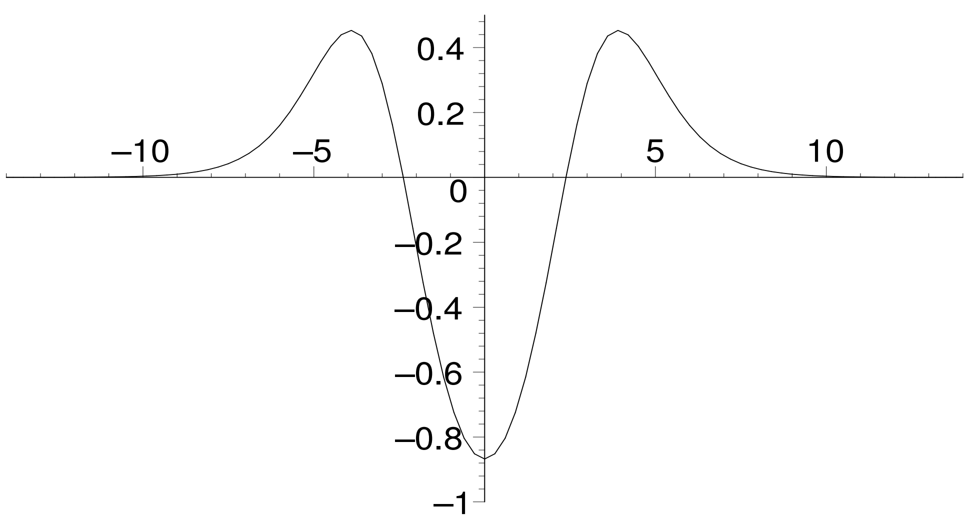


FIGURE 7. The graph of the eigenfunction $\Phi = \frac{dW}{d\gamma}$ corresponding to $\lambda = 0$ at the fold point $U(0)W(0) = 1.02$, $U'(\infty) = 1.347$.

So as A is increased above the threshold where the steady-state solution dissapears, the shape of the corresponding eigenfunction will deform the spike into two spikes. The resulting two spikes will then move away from each-other. If the distance between them becomes too big, their interaction becomes small enough and the whole process may repeat again.

10 Numerical Computations

We have performed numerical simulations of Gray-Scott model on a disk. We used a second order discretization in space, combined with the forward Euler method in time. Matlab was used for visualization. For all the simulations here, we chose $R = 3$, $\varepsilon = 0.05$, $\tau = 1$, and discretized the radial and anglular direction into 60 and 30 intervals, respectively. The time step was taken to be 0.00005. For initial conditions, we chose a ring of radius $r_0 = 1.5$ of width ε, and with very small, random perturbations in the angular direction.

Note that throughout this paper we have assumed that $\sqrt{6\varepsilon} \ll 1$, so that $\hat{A} \ll A$. (see (1.6)). Even assuming $\sqrt{6\varepsilon} = 0.1$, we would need to take much more than $600R$ mesh modes in the radial direction in order to resovle the ring whose core has width $O(\varepsilon)$. At this time, we do not have the code to accurately simulate this regime. Thus we cannot expect our simulation to have a good quantitative agreement with the theory.

Experiment 1. Here we qualitatively verify that the first unstable mode increases rapidly as A is increased (see Figure 2). Starting from the same initial condition of a ring of radius 1.5, Figure 8 shows the solution at time t=30 for $A = 0.8, 0.85, 0.87$ and 1.0. For these values of A, we observe that the ring breaks into m spots, where $m = 8, 11, 12, 13$, respectively. This agrees with our theoretical prediction that the first unstable mode is increased as A is increased.

Experiment 2. Next, we increase A, approaching the ring-splitting regime. Figure 9 shows a simulation for $A = 2.0$. The initial ring at $r_0 = 1.5$ starts to expand until its radius reaches about 2.25. It then breaks into many spots. This implies that all lower-modes are stable, but an instability at a very high mode is triggered. Moreover, the spots form both at the outside and at the inside of the ring.

Theoretically, $A = 2.0$ corresponds to the regime $\hat{A} \to \infty$. Note that for $R = 3$, we find from theory that r_R and A_c, defined in Conjecture 2.1, are $r_R = 2.238$ and $A_c = 1.837$, respectively. We cannot expect a good quantitative agreement since the value of ε in our simulations is not small enough; nevertheless, we do find that the ring breakup in our simulations occurs at about $r = 2.25$ which agrees well with the predicted value of $r_R = 2.238$.

Experiment 3 (Figure 10). For $A = 2.2$, the ring again expands until its radius reaches about 2.25. This time however, the the instability is such that the inside of the ring splits into many spots, whereas the outside of the ring moves towards to the boundary, and remains stable for a much longer time. Eventually however, the outside of the ring also breaks apart.

Experiment 4 (Figure 11). For $A = 2.5$, the ring splits into two. The two resulting rings then start travelling apart. Some time later, the inner ring breaks up. Then much later the outer ring also breaks.

Experiment 5 (Figure 12). Our last simulation is with $A = 4$. As a result, a single ring eventually splits into four. The resulting rings then lose their stability, one-by-one, starting from the innermost ring, and progressing

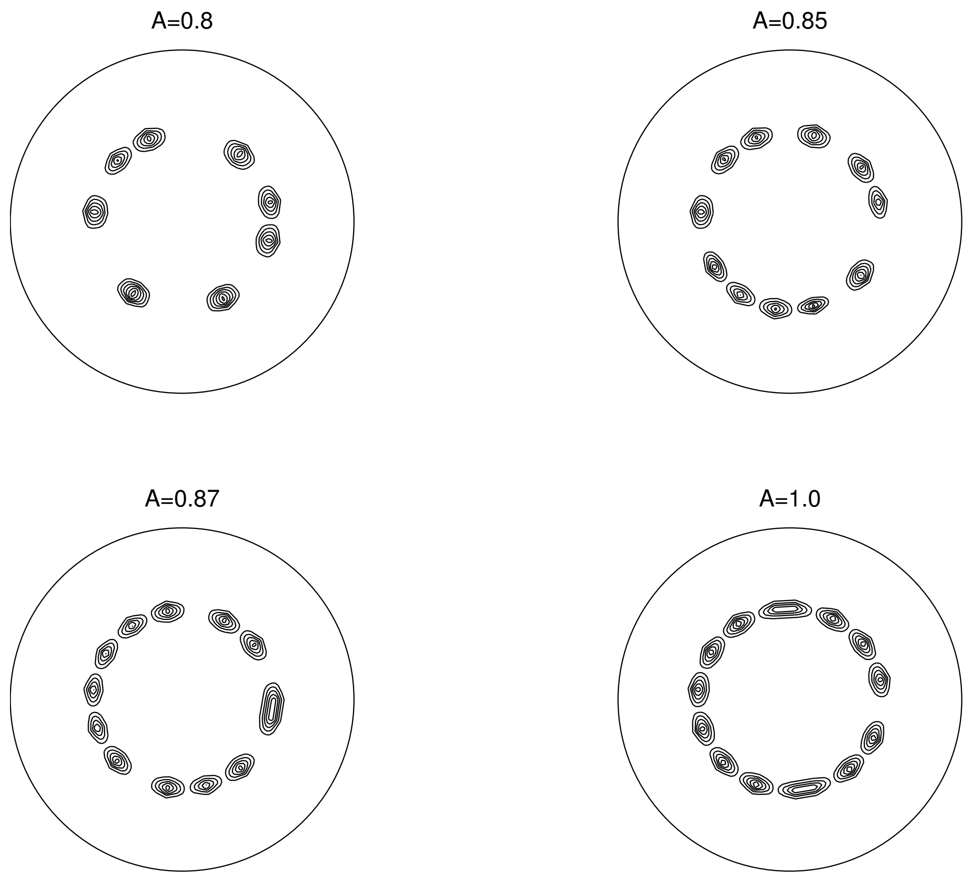


FIGURE 8. Solution at time $t = 30$ with A as indicated.

towards the outermost ring. Note however that the outer ring can remain stable for a very long time, and becomes unstable only after the adjacent ring has been broken up.

11 Conclusion and Discussion

We have performed an extensive study of the ring-like solutions for the Gray-Scott model in the regime where the ratio of the diffusivity coefficients $\frac{D_v^2}{D_u}$ is small, using both rigorous PDE theory (Section 3 to Section 8) and matched asymptotics approach (Section 9).

In Theorems 2.1, 8.1 and 8.3, we rigorously construct ring-like solutions in a ball, $\mathbb{R}^N$ an annulus, or the exterior of a ball. Our approach is a Liapunov-Schmidt reduction method, combined with asymptotic analysis. In the sub-regime where $\hat{A} = \frac{A}{\sqrt{6\varepsilon}}$ is of $O(1)$, we found that the ring-like solution on entire $\mathbb{R}^N$ exists only if $\hat{A}$ is below some threshold $\hat{A}_c$. This behaviour is very different from that of a bounded disk. For a disk of radius $R < \infty$, a ring-like solution always exists, and its radius is less than $r_R < R$, where r_R is a zero of $(J_1 J_{2,R})^{'}(r)$.

In Theorems 2.2 and 2.3, we study the stability of the ring-like solutions in $N = 2$. We have rigorously proved

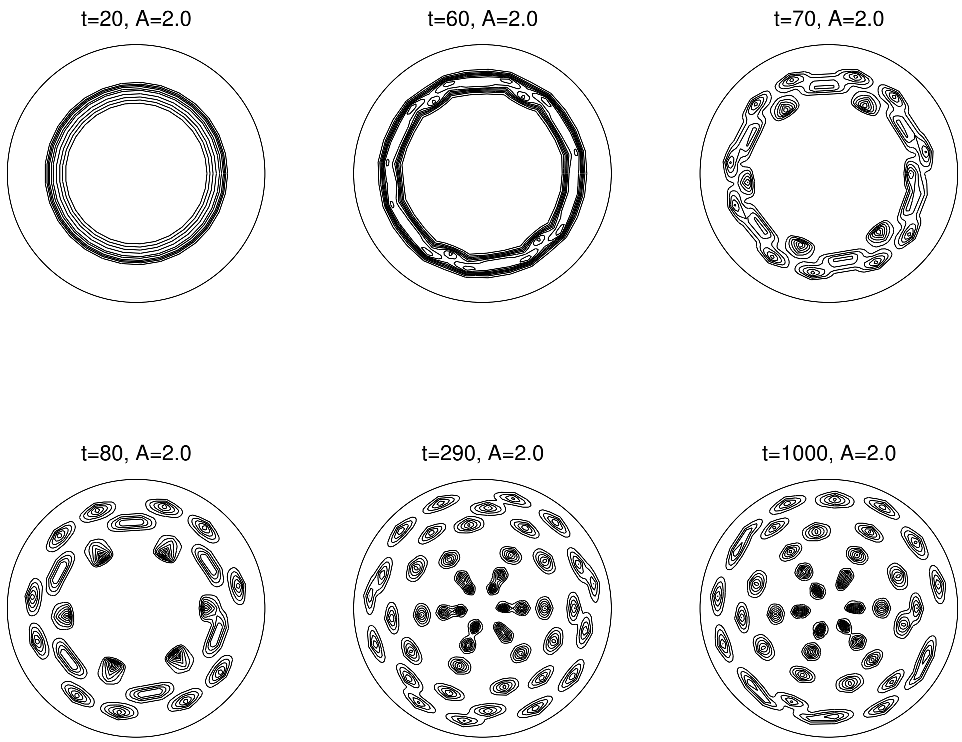


FIGURE 9. Contour plot of v for $A = 2.0$.

that the ring-like solutions are unstable for large modes $1 << m << \frac{1}{\epsilon}$. Our approach is based on the study of a nonlocal eigenvalue problem, using functional analysis developed in [**46**] and [**42**]. We found that the ring-like solution on the entire space is always unstable with respect to the first mode of the angular perturbation $\cos(\theta)$. By contrast, on a bounded domain, any given number of low modes can be made stable by choosing $\hat{A}$ big enough. However an open question remains: can a ring be stable with respect to *all* angular modes?

When $A = O(1)$, the equations for u and v cannot be decoupled at the core of the ring. In such a regime, questions about existence and stability of the ring are reduced to a one-dimensional core problem (9.1), which we can only solve numerically. Previous numerical studies of the core problem (see [**23**] and [**29**]) show that in one dimension, spike splitting will occur if A is increased beyond a certain threshold. In Section 9 we have found a similar threshold in two dimensions: if $A > A_c$ given by (2.21) then the ring will split into two rings. Numerical simulations confirm this result. For A just below A_c, we conjecture that there exists a ring solution whose radius is precisely r_R. The angular stability of such ring remains an open problem.

In this work we have addressed the breakup instability of a ring due to $O(1)$ perturbations in the profile. Such instabilities correspond to $O(1)$ unstable eigenvalues, and are referred to as the *large* eigenvalues. We have not addressed the *small* eigenvalues that arise due to translation invariance of the problem. An instablility of such

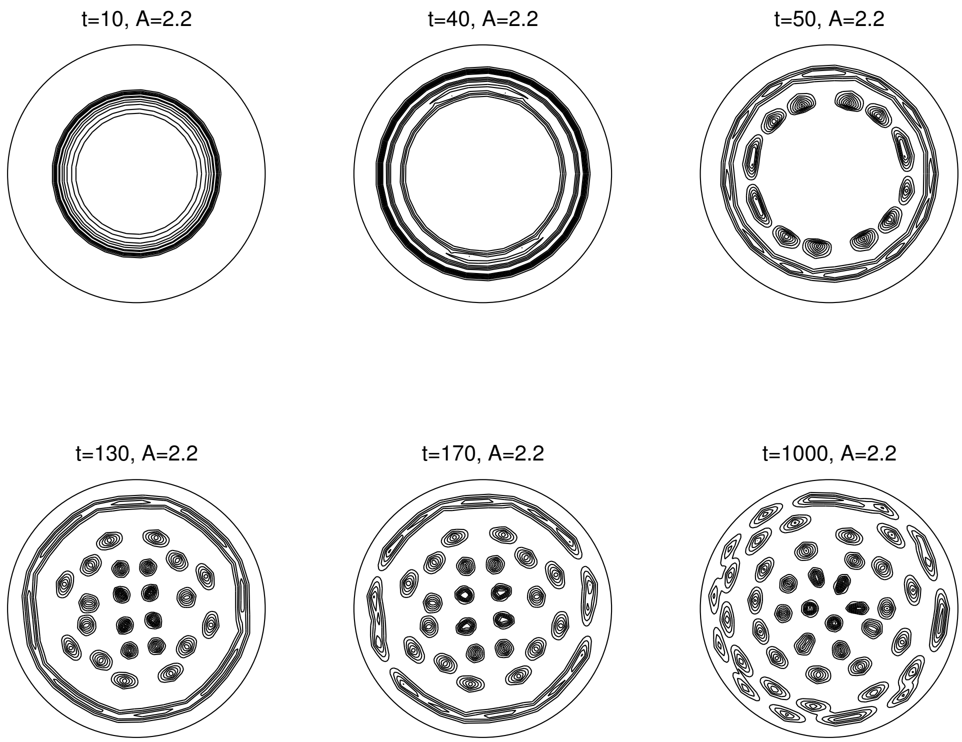


FIGURE 10. Contour plot of v for $A = 2.2$.

an eigenvalue can induce a dynamic change of the ring radius; higher mode small eigenvlaue can also lead to a zigzag-type instability. See [**24**] for the study on such an instability.

An interesting open problem is to examine a connection between a ring solution and a spot solution. From (8.10), by using the near-zero expansions of the Bessel functions, for a single ring on the entire domain with r small, we obtain:

$$\hat{A}^2 \sim \frac{9}{2} r \ln(r^{-1}), \qquad r \ll 1$$

However our analysis is only valid for $r \gg O(\varepsilon)$ and breaks down when $r = O(\varepsilon)$. On the other hand, it is easy to show that spot solutions exist in the regime $\hat{A}^2 = O\left(\varepsilon \ln(\varepsilon^{-1})\right)$. This suggests that a spot can bifurcate into a ring as $\hat{A}^2$ is slowly increased beyond $O\left(\varepsilon \ln(\varepsilon^{-1})\right)$. Indeed numerical simulations suggest that this is indeed the case. When A is large enough, an initial solution consisting of a spot tends to expand into a ring. The ring then continues to expand until it breaks up into spots. On the other hand, for smaller values of A, a single spot may be stable, or may undergo a self-replication, resulting in two or more spots.

We now compare the our results with those obtained by Morgan and Kaper in [**28**]. They used the following

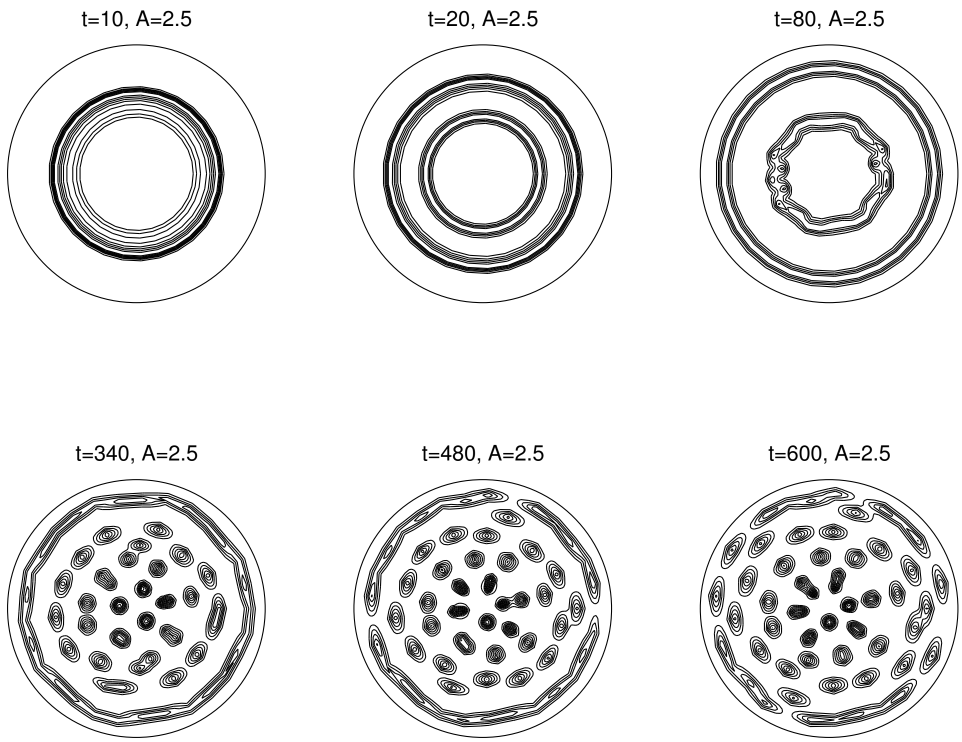


FIGURE 11. Contour plot of v for $A = 2.5$.

scaling of the Gray-Scott model:

$$\frac{dV(y,s)}{ds} = D\Delta V - BV + UV^2,$$
$$\frac{dU(y,s)}{ds} = \Delta U + A_{mk}(1-U) - UV^2.$$

By scaling the variables as follows:

$$V = \sqrt{A_{mk}}v, \quad U = u, \quad s = \frac{1}{B}t, \quad y = \frac{1}{\sqrt{A_{mk}}}x$$

we re-obtain our system (1.2) with

$$\tau = \frac{B}{A_{mk}}, \quad \varepsilon = \sqrt{\frac{DA_{mk}}{B}}, \quad A = \frac{\sqrt{A_{mk}}}{B}$$

or

$$A_{mk} = \frac{1}{\tau^2 A^2}, \quad B = \frac{1}{\tau A^2}, \quad D = \varepsilon^2 \tau.$$

The paper [**28**] obtains results for the ring location and its stability. In addition it contains a linear Turing analysis for the radially-symmetric solutions together with full numerical simulations.

For the location of the ring, in [**28**] the same formula (equation (2.35) of [**28**]) is obtained as in Theorem 2.1. They use a Melnikov-type calculation to obtain their results whereas we have used Lyapunov-Schmidt reduction.

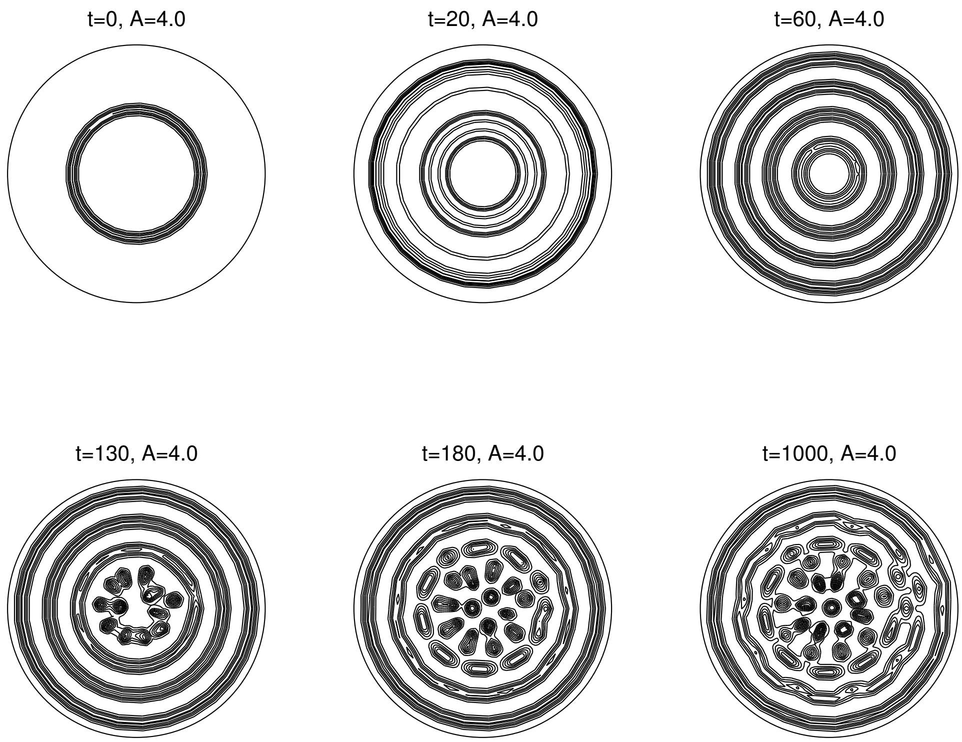


FIGURE 12. Contour plot of v for $A = 4.0$.

However, they do not have any analytical results on the existence of solution to equation (2.35) of [**28**]. Indeed, they consider only a bounded domain – in which case a ring solution exists for any choice of $\hat{A}$. Our result on existence of the bound $\hat{A}_c$ on $\hat{A}$ in case of the *unbounded* domain is new. In the case of the bounded domain, we rigorously show the existence of $r_R < R$, which has the property that the radius of the ring $r_0 \to r_R$ as $\hat{A} \to \infty$. This is also a new result. Another new result that we have obtained using the comparison principle, is the existence of $\hat{A}$ for any given ring radius r_0 in the case of the unbounded domain. We also consider the general N-dimensional case and examine the qualitative differences between two and higher dimensions (see the remark after proof of Lemma 8.1). The analysis of [**28**] on the other hand, is restricted to two dimensions.

For the stability analysis with respect to angular perturbations, we obtain a simple sufficient condition (see Theorem 2.2) for when the ring is unstable with respect to mode m. This condition is also necessary when $\tau = 0$. Our condition involves only Bessel functions of order m and the ring radius r_0. Our proof is rigorous and involves no numerical computations. By contrast, the stability criteria for the m-th mode in [**28**] is implicitly contained in integrals of hypergeometric functions which is then solved by Mathematica. In both cases, the stability analysis is reduced to a non-local eigenvalue problem. We also find that all modes $1 \ll m \ll O(\frac{1}{\varepsilon})$ are unstable for any $\tau \leq O(1)$, and all modes $m > \frac{\sqrt{5}}{2}\frac{r_0}{\varepsilon}$ are stable (Theorem 2.3). These results are new.

For the ring-splitting regime $A = O(1)$, we use formal asymptotics and one-dimensional numerics to derive an explicit bound A_c in terms of R (see (2.21)) such that the ring-splitting occurs when $A > A_c$. This is a new result. We also show that the radius of the ring for A just below A_c is precisely r_R. We then use full numerical simulation to confirm existence of the ring-splitting regime. In [**28**], ring splitting is observed numerically but no analysis of this regime is performed there.

Finally, we mention related works by Muratov, Osipov and Kerner [**29**], [**30**], [**31**], [**22**]. In [**29**] the authors use formal asymptotics to derive an expression for the radius of the ring in three dimensions. Their result (see (5.12) of [**29**]) does not depend on Bessel functions, unlike the result obtained here. In [**31**] they also numerically observe ring breakup into spots, but no rigorous analysis is provided. In [**22**], a general, qualitative mechanism of ring breakup into spots for general reaction diffusion systems is also discussed (see pp. 430-433), and a scaling law is derived, predicting the instability of some high modes m, but without giving bounds on the instability band. Theorems 2.2, 2.3 provide these bounds in the specific case of the Gray-Scott model.

Appendix A: Proof of Proposition 5.1

We first prove that if (ϕ, c) satisfy (5.5) and (5.6), then for ε sufficiently small, we have

$$\|\phi\|_* \leq C\|h\|_*. \tag{A.1}$$

We prove it by contradiction. Suppose not. Then there exists a sequence $\varepsilon_k \to 0$ and a sequence of functions ϕ_{ε_k} satisfying (5.5)-(5.6) such that the following holds:

$$\|\phi_{\varepsilon_k}\|_* = 1, \|h_k\|_* = o(1), \int_{I_{\varepsilon_k}} \phi_{\varepsilon_k} Z_{\varepsilon_k,t}(t+\varepsilon_k y)^{N-1} = 0.$$

For simplicity of notations, we drop the dependence on k. Let

$$L_\varepsilon[\phi] = \phi^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi^{'} - \phi + 2w_{\varepsilon,t}\phi - 2(1-\xi_\varepsilon)\frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi}{\int_{I_\varepsilon} w_{\varepsilon,t}^2} w_{\varepsilon,t}^2. \tag{A.2}$$

Then ϕ_ε satisfies

$$L_\varepsilon[\phi_\varepsilon] = h + cZ_{\varepsilon,t}, \quad \phi_\varepsilon^{'}(-\frac{t}{\varepsilon}) = \phi_\varepsilon^{'}(\frac{R-t}{\varepsilon}) = 0. \tag{A.3}$$

Multiplying (A.3) by $w_{\varepsilon,t}^{'}$ and integrating over I_ε, we obtain that

$$c\int_{I_\varepsilon} Z_{\varepsilon,t} w_{\varepsilon,t}^{'} = -\int_{I_\varepsilon} h w_{\varepsilon,t}^{'} + \int_{I_\varepsilon} (L_\varepsilon[\phi_\varepsilon]) w_{\varepsilon,t}^{'}. \tag{A.4}$$

The left hand side of (A.4) is simply $c(\int_R p w^{p-1}(w^{'})^2 + o(1))$ since $Z_{\varepsilon,t} = 2w_{\varepsilon,t}w_{\varepsilon,t}^{'} + O(\varepsilon)$. The first term on the right hand side of (A.4) can be estimated as

$$\int_{I_\varepsilon} h w_{\varepsilon,t}^{'} = O(\|h\|_*).$$

The last term equals

$$\int_{I_\varepsilon} (L_\varepsilon[\phi_\varepsilon]) w_{\varepsilon,t}^{'} = \int_{I_\varepsilon} \left[\phi_\varepsilon^{''} + \frac{\varepsilon}{t+\varepsilon y}\phi_\varepsilon^{'} - \phi_\varepsilon + 2w_{\varepsilon,t}\phi_\varepsilon\right] w_{\varepsilon,t}^{'} - 2(1-\xi_\varepsilon)\frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi_\varepsilon}{\int_{I_\varepsilon} w_{\varepsilon,t}^2}\int_{I_\varepsilon} w_{\varepsilon,t}^2 w_{\varepsilon,t}^{'}$$

$$= o(\|\phi\|_*).$$

Hence we obtain that

$$|c| = O(\|h\|_*) + o(\|\phi\|_*), \quad \|h + cZ_{\varepsilon,t}\|_* = o(1). \tag{A.5}$$

Next we claim that $|\phi_\varepsilon(y)| \to 0$ in any compact interval of R. In fact, we consider $\bar{\phi}_\varepsilon(y) = \phi_\varepsilon \chi(\varepsilon y)$. Then it is easy to see that $\|\bar{\phi}_\varepsilon\|_{H^2} \leq C$ and hence $\bar{\phi}_\varepsilon \to \phi_0$ weakly in $H^2(R)$ and ϕ_0 satisfies

$$L\phi_0 = 0, |\phi_0| \leq Ce^{-\mu_1|y|}.$$

By Lemma 3.1, we must have $\phi_0 = cw^{'}$. On the other hand, $\int_{I_\varepsilon} \phi_\varepsilon Z_{\varepsilon,t}(t+\varepsilon y)^{N-1} dy = 0$ and hence $\int_R \phi_0 w^{p-1} w^{'} = 0$, which implies that $c = 0$. Hence, $\phi_\varepsilon \to 0$ in any compact interval of R. This shows that

$$\|w_{\varepsilon,t}\phi_\varepsilon\|_* = \sup_{y \in I_\varepsilon} |w_{\varepsilon,t}(y)\phi_\varepsilon(y)| = o(1). \tag{A.6}$$

On the other hand, by Lebesgue's Dominated Convergence Theorem, we have that

$$\int_{I_\varepsilon} w\phi_\varepsilon (t+\varepsilon y)^{N-1} \to 0,$$

which implies that

$$\| \frac{\int_{I_\varepsilon} w_{\varepsilon,t}\phi_\varepsilon}{\int_{I_\varepsilon} w_{\varepsilon,t}^2} w_{\varepsilon,t}^2 \|_* = o(1). \tag{A.7}$$

Thus we have arrived at the following situation: ϕ_ε satisfies

$$\phi_\varepsilon^{''} + \frac{\varepsilon(N-1)}{t+\varepsilon y}\phi_\varepsilon^{'} - \phi_\varepsilon = o(1), \quad \phi_\varepsilon^{'}(-\frac{t}{\varepsilon}) = \phi_\varepsilon^{'}(\frac{R-t}{\varepsilon}) = 0, \quad \phi_\varepsilon = O(1). \tag{A.8}$$

We claim first that $\phi_\varepsilon(0) = o(1)$. In fact, suppose not. There exists a sequence of $\varepsilon_k \to 0$ such that $\phi_\varepsilon(0) \geq \delta_0$ for some constant $\delta_0 > 0$. By taking a subsequence, $\phi_{\varepsilon_k}(y) \to \phi_0(y)$ in $C^2_{loc}(R)$ and ϕ_0 satisfies

$$\phi_0^{''} - \phi_0 = 0, \quad \phi_0(0) \geq \delta_0 > 0, \quad \phi_0 = O(e^{-\mu_1 <y>})$$

which is clearly impossible.

So $\phi_\varepsilon(0) = o(1)$. Similarly we have $\phi_\varepsilon^{'}(0) = o(1)$. Then by the comparison principle, $\phi_\varepsilon = o(1)$ for $y \in I_\varepsilon$.

This proves (5.7).

Finally, the existence follows from the Fredholm alternative. To this end, let us set

$$\mathcal{H} = \{u \in H^1(\mathbb{R}^N) | (u, w_{\varepsilon,t}^{'}) = 0\}.$$

Observe that ϕ solves (5.5) and (5.6) if and only if $\phi \in H^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} (\nabla\phi\nabla\psi + \phi\psi) - p < w_{\varepsilon,t}^{p-1}\phi, \psi >_\varepsilon - qm \frac{\int_{I_\varepsilon} w_{\varepsilon,t}^{m-1}\phi}{\int_{I_\varepsilon} w_{\varepsilon,t}^m} < w_{\varepsilon,t}^p, \psi >_\varepsilon = < h, \psi >_\varepsilon, \quad \forall \psi \in H^1(\mathbb{R}^N)$$

This equation can be rewritten in the following form

$$\phi + \mathcal{S}(\phi) = \bar{h}, \tag{A.9}$$

where $\mathcal{S}$ is a linear compact operator form $\mathcal{H}$ to $\mathcal{H}$, $\bar{h} \in \mathcal{H}$ and $\phi \in \mathcal{H}$.

Using the Fredholm alternative, in order to show that equation (A.9) has a uniquely solvable solution for each $\bar{h}$, it is enough to show that the equation has a unique solution for $\bar{h} = 0$. To this end, we assume the contrary. That is, there exists (ϕ, c) such that

$$L_\varepsilon[\phi] = cZ_{\varepsilon,t}, \tag{A.10}$$

$$\phi^{'}(-\frac{t}{\varepsilon}) = 0, \phi(y) \to 0 \text{ as } y \to +\infty, < \phi, Z_{\varepsilon,t} >_\varepsilon = 0. \tag{A.11}$$

From (A.10), it is easy to see that $\|\phi\|_* < +\infty$. So without loss of generality, we may assume that $\|\phi\|_* = 1$. But then this contradicts to (A.1).

□

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