The Existence and Stability of Spike Equilibria in the One-Dimensional Gray-Scott Model on a Finite Domain

Theodore Kolokolnikov*, Michael J. Ward†, Juncheng Wei‡

Abstract

Results for the existence, stability, and pulse-splitting behavior of spike patterns in the one-dimensional Gray-Scott model on a finite domain in the semi-strong spike-interaction regime are summarized. Conditions on the parameters for the existence of competition instabilities, synchronous oscillatory instabilities, or pulse-splitting behavior of spike patterns are given.

Keywords: spike patterns, nonlocal eigenvalue problem, oscillatory instability, pulse-splitting.

1 Introduction

In this letter we summarize some results of [1] and [2] for the existence and stability of spike patterns in the one-dimensional Gray-Scott (GS) reaction-diffusion model. The GS model, first introduced in [3], and studied numerically in [4], can be written in dimensionless form as

$$v_t = \varepsilon^2 v_{xx} - v + Auv^2 \,, \qquad -1 < x < 1 \,, \quad t > 0 \,; \qquad v_x = 0 \,, \quad x = -1, 1 \,, \tag{1.1a}$$

$$\tau u_t = Du_{xx} + (1-u) - uv^2 \qquad -1 < x < 1 \,, \quad t > 0 \,; \qquad u_x = 0 \,, \quad x = -1, 1 \,. \tag{1.1b}$$

Here $A > 0$, $D > 0$, $\tau > 1$, and $\varepsilon \ll 1$. This form of the GS model was first introduced in [5].

We consider (1.1) in the semi-strong spike interaction regime defined by $D = O(1)$ and $\varepsilon \ll 1$. In this limit, singular perturbation techniques can be used to partially decouple u and v. From a mathematical viewpoint, our analysis provides a careful case study for the long-term goal of classifying generic mechanisms leading to instabilities of spike-type solutions in reaction-diffusion systems. For general reaction-diffusion systems in the weakly nonlinear regime, a similar classification based on amplitude equations and normal forms has been undertaken over the past several decades. As a step

*Department of Mathematics, University of British Columbia, Vancouver, Canada V6T 1Z2
†Department of Mathematics, University of British Columbia, Vancouver, Canada V6T 1Z2, (corresponding author)
‡Department of Mathematics, Chinese University of Hong Kong, New Territories, Hong Kong

towards a classification for instabilities of spike solutions, our results (cf. [1], §2) show that certain instabilities in the GS model have a direct spectral equivalence with those in the Gierer-Meinhardt model of [6] derived in [7] in the semi-strong interaction limit.

For the infinite domain problem, the existence and stability of a one-spike solution for the GS model in the semi-strong interaction regime was studied analytically in [8], [9], [10], and [5]. Periodic solutions were also constructed in [8]. A numerical example of pulse-splitting with a qualitative analysis of the dynamical mechanism of pulse-splitting is given in [11].

The weak spike interaction regime is defined by $D = O(\varepsilon^2)$, and $\varepsilon \ll 1$. In this latter regime, where u and v are both localized near certain points in the domain, the pioneering works of [12] and [13] showed that both pulse-splitting behavior and spatio-temporal chaos is possible. In terms of numerically computed bifurcation diagrams, a clear mechanism for the occurrence of pulse-splitting was formulated. General criteria for pulse-splitting are given in [14]. For symmetric initial data, the pulse-splitting that occurs in the weak interaction regime exhibits edge-splitting, whereas in the semi-strong interaction regime pulses split roughly simultaneously.

In the semi-strong regime where $D = O(1)$, and $\varepsilon \ll 1$, there are three main parameter regimes for A where different behaviors occur. In §2, the results of [1] are summarized for the *low-feed rate* regime, $A = O(\varepsilon^{1/2})$, where both competition and oscillatory instabilities can occur. These results are new. Some new results of [2] for the pulse-splitting regime, $A = O(1)$, verifying the criteria of [14] and predicting the number of pulse-splitting events, are given in §3. In the intermediate regime $O(\varepsilon^{1/2}) \ll A \ll 1$, our results are rather closely related to those in [8], [9], and [10]. For this regime, we refer to §3.4 and §5 of [1] for a detailed comparison to these previous works.

2 The Low-Feed Rate Regime: $\mathcal{A} = O(\varepsilon^{1/2})$

In the low feed-rate regime, we define a new $O(1)$ variable $\mathcal{A}$ by $A = \varepsilon^{1/2}\mathcal{A}$. The first result of [1] for this regime characterizes symmetric k-spike equilibrium patterns, characterized by spikes in v that have a common amplitude. For each $k = 1, 2 \ldots$, and for $\mathcal{A} > \mathcal{A}_{ke}$, there are two branches of such solutions for (1.1) that meet at the saddle-node bifurcation value $\mathcal{A}_{ke}$ given by

$$\mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{2.1}$$

These solution branches are conveniently parameterized in terms of s, with $0 < s < \infty$, where

$$s \equiv \frac{1 - U_\pm}{U_\pm}\,, \qquad U_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\,\right]\,. \tag{2.2}$$

We refer to the range $0 < s < 1$ as the *large solution branch* ($U = U_+$ is large), while the range $s > 1$ is called the *small solution branch* ($U = U_-$ is small). The precise result is as follows (see Proposition 2.1 of [1]):

Proposition 1: *Let $\varepsilon \to 0$, with $\mathcal{A} = O(1)$ and $D = O(1)$. Then, when $\mathcal{A} > \mathcal{A}_{ke}$, the small and large k-spike symmetric equilibrium solutions for (1.1), denoted by u_-, v_- and u_+, v_+, respectively, are given asymptotically by*

$$v_\pm(x) \sim \frac{1}{\sqrt{\varepsilon}\mathcal{A}U_\pm} \sum_{j=1}^{k} w\left[\varepsilon^{-1}(x - x_j)\right] , \qquad u_\pm(x) \sim 1 - \frac{(1 - U_\pm)}{a_g} \sum_{j=1}^{k} G(x; x_j) . \tag{2.3}$$

Here $x_j = -1 + \frac{(2j-1)}{k}$, $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, $w(y) = \frac{3}{2} sech^2(y/2)$ satisfies $w'' - w + w^2 = 0$, and $G(x; x_j)$ is the Green's function satisfying $DG_{xx} - G = -\delta(x - x_j)$ with $G_x(\pm 1; x_j) = 0$.

To analyze the stability of these equilibrium solutions on an $O(1)$ time-scale, we let $u(x,t) = u_\pm(x) + e^{\lambda t}\eta(x)$, and $v(x,t) = v_\pm(x) + e^{\lambda t}\phi(x)$, where ϕ is a localized eigenfunction of the form

$$\phi(x) \sim \sum_{j=1}^{k} c_j \Phi\left[\varepsilon^{-1}(x - x_j)\right] , \tag{2.4}$$

for some c_j to be found. We consider eigenfunctions with $\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy \neq 0$. In §3.1 of [1], the following spectral problem for $\Phi(y)$ is derived (see Proposition 3.2 of [1]):

Proposition 2: *Assume that $0 < \varepsilon \ll 1$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues determining the stability of k-spike equilibria satisfy the nonlocal eigenvalue problem*

$$L_0\Phi - \chi_{gs} w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy} \right) = \lambda\Phi , \qquad -\infty < y < \infty ; \qquad \Phi \to 0 , \quad \text{as} \quad |y| \to \infty . \tag{2.5}$$

Here the operator L_0 is $L_0\Phi \equiv \Phi'' - \Phi + 2w\Phi$, and the multiplier $\chi_{gs} = \chi_{gs}(z; j)$ is defined by

$$\chi_{gs} \equiv 2s \left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)} \left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)} \right] \right)^{-1} , \tag{2.6}$$

where $z \equiv \tau\lambda$, $\theta_\lambda \equiv \theta_0\sqrt{1+z}$, and $\theta_0 \equiv D^{-1/2}$. Here s is given in (2.2). The coefficients $\boldsymbol{c}^t = (c_1, \ldots, c_k)$ in (2.4) are given by

$$\boldsymbol{c}_1^t = \frac{1}{\sqrt{k}}(1, \ldots, 1) ; \quad c_{l,j} = \sqrt{\frac{2}{k}} \cos\left(\frac{\pi(j-1)}{k}(l - 1/2) \right) , \quad j = 2, \ldots, k . \tag{2.7}$$

There is an equivalent formulation of (2.5) which states that the eigenvalues of (2.5) with $\int_{-\infty}^{\infty} w\Phi\, dy \neq 0$ are the union of the zeros of the functions $g_j(\lambda) = 0$ for $j = 1,..,k$, where

$$g_j(\lambda) \equiv C_j(\lambda) - f(\lambda)\,, \qquad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\,(L_0-\lambda)^{-1}\, w^2\, dy}{\int_{-\infty}^{\infty} w^2\, dy}\,, \qquad C_j(\lambda) \equiv \left[\chi_{gs}\right]^{-1}. \tag{2.8}$$

By analyzing the zeroes of $g_j(\lambda)$ in the right half-plane $\text{Re}(\lambda) \geq 0$, the main stability result for multi-spike solutions is as follows (cf. Proposition 3.10 and 3.13 of [1]):

Proposition 3: *The large solution u_+, v_+ is unstable for any $0 < s < 1$, $k \geq 1$, and $D > 0$. Next, let $k > 1$, and consider the multi-spike small solution u_-, v_-, where $s > 1$. For $\mathcal{A} > \mathcal{A}_{kL}$, the solution will be stable on an $O(1)$ time-scale when $0 < \tau < \tau_{hL}$. Alternatively, on the range $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$, the small solution is unstable for any $\tau > 0$. The threshold $\mathcal{A}_{kL}$ is given analytically by*

$$\mathcal{A}_{kL} = \frac{\mathcal{A}_{ke}\left((\gamma_k/2) + 2\sinh^2(\theta_0/k)\right)}{\left(\left[(\gamma_k/2) + 2\sinh^2(\theta_0/k)\right]^2 - (\gamma_k/2)^2\right)^{1/2}}\,, \qquad \gamma_k \equiv 1 + \cos\left(\frac{\pi}{k}\right). \tag{2.9}$$

Let $\mathcal{A}$ satisfy $\mathcal{A} > \mathcal{A}_{kL}$. Then, as τ increases beyond τ_{hL}, a Hopf bifurcation in the spike amplitudes was computed numerically in [1]. The threshold τ_{hL} is given by the *minimum value* of the set τ_j, $j = 1,..,k$, for which $g_j(\lambda) = 0$, $j = 1,..,k$, has complex conjugate roots on the imaginary axis. Let $\lambda = \pm i\lambda_h$ be the corresponding value of λ. Then, as was shown in [1], the unstable eigenfunction generically has the form of a *synchronous oscillatory instability* with

$$v = v_- + \delta e^{i\lambda_h t}\phi + \text{c.c}\,, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x-x_n)\right]\,, \quad c_n = 1\,, \quad n = 1,..,k\,. \tag{2.10}$$

Here c.c denotes complex conjugate and $\delta << 1$. Alternatively, suppose that $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$. Then, for any $\tau > 0$, the dominant initial instability was shown in [1] to have the form

$$v = v_- + \delta e^{\lambda_{Rk} t}\phi\,, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x-x_n)\right]\,, \quad c_n = \cos\left(\frac{\pi(k-1)}{k}(n-1/2)\right)\,, \quad n = 1,..,k\,. \tag{2.11a}$$

Here $\delta \ll 1$, and $\lambda_{Rk} > 0$ is the unique root of $g_k(\lambda_R) = 0$. Since $\sum_{n=1}^{k} c_n = 0$, this instability locally conserves the sum of the heights of the spikes. Hence, it is referred to as a *competition instability.* The numerical experiments in §3.3 of [1] show that this instability leads to a spike competition process whereby certain spikes in a spike sequence are ultimately annihilated. Finally, in Proposition 3.3 of [1] it was shown there there is a spectral equivalence principle between (2.5) and a corresponding nonlocal eigenvalue problem derived in Proposition 2.3 of [7] for the Gierer-Meinhardt (GM) model.

3 The Pulse-Splitting Regime: $A = O(1)$

In this regime, where $A = O(1)$, it was shown in [2] that equilibrium k-spike solutions can be constructed in terms of the solutions $V(y) > 0$ and $U(y) > 0$ to a certain *core problem* defined by

$$V'' - V + V^2 U = 0\,, \qquad U'' = UV^2\,, \qquad 0 < y < \infty\,, \tag{3.1a}$$

$$V'(0) = U'(0) = 0\,; \quad V \to 0\,, \quad U \sim By\,, \quad \text{as} \quad y \to \infty\,; \quad B \equiv A \tanh(\theta_0/k)\,. \tag{3.1b}$$

Here $\theta_0 \equiv D^{-1/2}$. This core problem, without the term $B \equiv A \tanh(\theta_0/k)$ associated with the finite domain, was first derived in [5] for the infinite-line problem. The equilibrium result given in Proposition 3.1 of [2] is as follows:

Proposition 4: *Let $\varepsilon \to 0$, $A = O(1)$, $\varepsilon A/\sqrt{D} \ll 1$, and suppose that (3.1) has a solution. Then, the v-component for a k-spike equilibrium solution to (1.1) is given by*

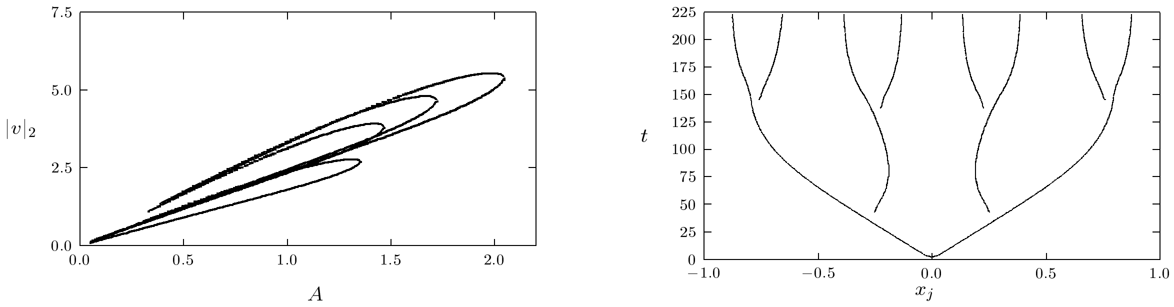
$$v \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} \left(V\left[\varepsilon^{-1}(x - x_j)\right] + O\left(\frac{\varepsilon A}{\sqrt{D}}\right) \right)\,. \tag{3.2}$$

In the j^{th} inner region, where $|x - x_j| = O(\varepsilon)$, u satisfies $u \sim \frac{\varepsilon}{A\sqrt{D}} U\left[\varepsilon^{-1}(x - x_j)\right]$. The outer solution for u, valid for $|x - x_j| \gg O(\varepsilon)$, and $j = 1, \ldots, k$, has the form $u \sim 1 - \frac{1}{a_g} \sum_{j=1}^{k} G(x; x_j)$, where a_g, x_j, and $G(x; x_j)$ are as defined in Proposition 1.

The core problem (3.1) was studied qualitatively and numerically in [2] in terms of B. In [2] it was shown that $0 < \gamma < \frac{3}{2}$, where $\gamma \equiv U(0)V(0)$. Numerical solutions to (3.1) were computed for which V has a single maximum at $y = 0$ as $\gamma \to 3/2$ from below, and a resulting curve $B = B(\gamma)$ was computed. This limiting solution for V asymptotically matches onto the solution constructed in the low feed-rate regime in §2. As shown in [2], the curve $B = B(\gamma)$ is double-valued with $B \to 0$ as $\gamma \to 0$ and as $\gamma \to 3/2$, and it has a saddle-node bifurcation point at the maximum value B_c of B given by $B_c = 1.347$ (see also [5]), where $\gamma = \gamma_c = 1.02$. We refer to the range $\gamma_c < \gamma < 3/2$ and $0 < \gamma < \gamma_c$ as the *primary* and *secondary* branches of the $B = B(\gamma)$ bifurcation diagram. Since $B \equiv A \tanh(\theta_0/k)$ from (3.1b), these results of [2] show that equilibrium k-spike solutions exist only when A is small enough. In particular, we have (cf. Proposition 3.2 of [2]):

Proposition 5: *Let $\varepsilon \ll 1$, $A = O(1)$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, there will be no k-spike equilibrium solution to (1.1) that merges onto the low feed-rate regime solution when*

$$A > A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right)\,. \tag{3.3}$$



(a) $|v|_2$ versus A

(b) t versus x_j

Figure 1: Left figure: bifurcation diagram when $A = O(1)$ for k-spike solutions with $\varepsilon = 0.01$, $D = 0.1$, and $k = 1, \ldots, 4$. The saddle-node values A_{pk} increase with k. Right figure: trajectories of the maxima of v showing pulse-splitting behavior when $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and $\tau = 2.0$.

Next, the following three pulse-splitting criteria of [14] were verified in [1]: 1) each k-spike equilibrium branch must have a saddle-node bifurcation that occurs at approximately the same bifurcation value (i.e. the *lining-up property*): 2) for each k, one branch of equilibria is unstable while the other is stable: 3) the linearization around the equilibrium solution at each saddle-node value has a *dimple-shaped* eigenfunction associated with a zero eigenvalue. When these criteria of [14] are satisfied, and when the bifurcation parameter is taken to be only slightly beyond the saddle-node value, pulse-splitting should occur from a single localized initial pulse due to the *ghost* of the dimple eigenfunction. In Fig. 1(a) we plot the norm $|v|_2$, defined by $|v|_2^2 \equiv \int_{-1}^{1} v^2\, dx \sim \varepsilon^{-1} k D \int_{-\infty}^{\infty} V^2\, dy$, versus A when $D = 0.1$, to show that the approximate lining-up condition 1) holds. For $\tau \ll O(\varepsilon^{-1})$ the stability condition 2) of [14] is verified in §5 and §6 of [2] by showing that the primary and secondary branches are stable and unstable, respectively. Finally, the dimple eigenfunction property is verified in §6 of [2]. Therefore, we conjecture:

Conjecture 1: *Let $\varepsilon \ll 1$, $\tau \ll O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Suppose that we have even one-spike initial data centered at the origin. Then, the final equilibrium state is stable, and it has 2^m spikes where, for some integer $m \geq 0$, A is related to A_{pk} by*

$$A_{p2^{m-1}} < A < A_{p2^m}\,. \tag{3.4}$$

To verify this conjecture, we computed numerical solutions to (1.1) starting from an initial pulse in v centered at $x = 0$. In Fig. 1(b) we show pulse-splitting behavior for $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and

$\tau = 2.0$. For these values we calculate $A_{p4} = 2.045 < A < A_{p8} = 3.583$. Thus, (3.4) correctly predicts the eight-spike final state in Fig. 1(b). In addition, in Experiment 5 of [2], Conjecture 1 correctly predicts the final eight-spike solution for the pulse-splitting example in Fig. 4 of [11].

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