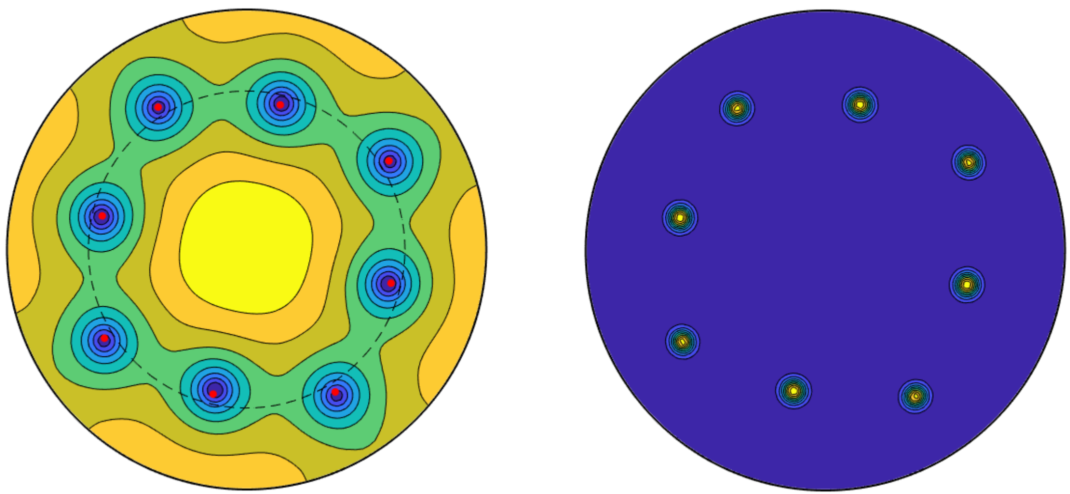
A ring of spikes and its stability



Introduction

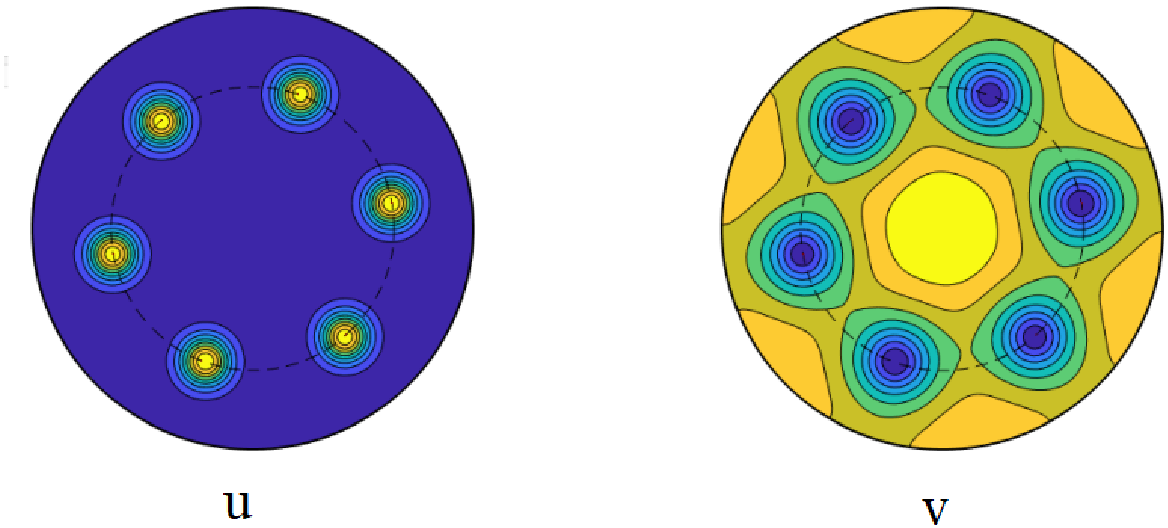
- Schnakenberg model:

$$u_t = \varepsilon^2 \Delta u - u + u^2 v, \qquad 0 = \Delta v + A - u^2 v \frac{1}{\varepsilon^2} \frac{1}{\log \varepsilon^{-1}} \tag{1}$$

  with the usual Neumann boundary conditions inside a radially symmetric domain Ω_b, which we take to be either a disk or an annulus:

$$\Omega_b = \{x : b < |x| < 1\} \, . \tag{2}$$

- Example of ring solution:



Reduced equations for spike motion

- Define

$$\begin{cases} \Delta G - \frac{1}{\pi} = -\delta(x-\xi), \quad x, \xi \in \Omega_b, \\ \partial_n G = 0, \quad x \in \partial\Omega_b, \\ \int_{\Omega_b} G(x,\xi)dx = 0 \end{cases} \tag{3}$$

- Spike motion:

$$\frac{dx_k}{dt} \sim -\frac{\varepsilon^2}{\eta} \frac{2}{\int w^2} S_k \sum_{j=1}^{N} S_j \nabla G_{kj}; \tag{4}$$

$$\sum_{j=1}^{N} S_j = |\Omega|\, A; \quad \frac{\eta \int w^2}{S_k} = T - \sum_{j=1}^{N} S_j G_{kj}. \tag{5}$$

$$G_{kj} = \begin{cases} G(x_k, x_j), & \text{if } k \neq j \\ \frac{1}{2\pi} \log \varepsilon^{-1} + H(x_j, x_j), & \text{if } k = j \end{cases}, \quad \nabla G_{kj} = \begin{cases} \nabla_{x_k} G(x_k, x_j), & \text{if } k \neq j \\ \nabla_x H(x,\xi)|_{\substack{x=x_k \\ \xi = x_j}}, & \text{if } k = j \end{cases} \tag{6}$$

$$H = G + \frac{1}{2\pi} \log|x - \xi|$$

and define

Spike ring radius

- Let

$$J(r,R,l) = \begin{cases} G(r, Re^{i2\pi l/N}), & \text{if } l \neq 0 \pmod N \\ \frac{1}{2\pi}\log \varepsilon^{-1} + H(r,R), & \text{otherwise} \end{cases} \tag{7}$$

- Then ring radius R satisfies

$$\sum_{k=0}^{N-1} J_r(R,R,k) = 0. \tag{8}$$

- Explicit computation for a unit disk:

$$R^2 - \frac{1}{2} + \frac{1}{2N} + \frac{1}{R^{-2N}-1} = 0. \tag{9}$$

Ring radius R for a ring of N spikes

N	2	3	4	5	6	7	8	9	10
R	0.4536	0.5517	0.5985	0.6251	0.6417	0.6527	0.6604	0.6662	0.6706

(10)

- For an annulus $b < |x| < 1$:

$$0 = \frac{R^2 - b^2}{(1-b^2)} - \frac{1}{2} + \frac{1}{2N} + \sum_{p=0}^{\infty}\left\{\frac{b^{2Np}}{R^{-2N} - b^{2Np}} - \frac{b^{2N(p+1)}}{R^{2N} - b^{2N(p+1)}}\right\}. \tag{11}$$

Stability, LARGE eigenvalues:

- Previous result (for ***any***) configuration:

$$A_{l0} \sim \frac{1}{\log \varepsilon^{-1}} \frac{N}{|\Omega|} \left(2\pi \int w^2 \right)^{1/2}. \tag{12}$$

- ***More accurate:*** Define

$$\Upsilon(m) := \sum_{l=0}^{N-1} \exp(2\pi l m i/N) J(r, R, l); \quad \text{and} \quad \tilde{\Upsilon} := \Upsilon - \frac{1}{2\pi} \log \varepsilon^{-1};$$

Define

$$A_l(m) = \frac{1}{\log \varepsilon^{-1}} \frac{N}{|\Omega|} \left(2\pi \int w^2 \right)^{1/2} \left(1 + \frac{2\pi}{\log \varepsilon^{-1}} \tilde{\Upsilon}(m) \right)^{-1/2}; \tag{13}$$

$$A_l = \max_m A_l(m) = A_l(\lfloor N/2 \rfloor) \tag{14}$$

For even N,

$$A_l = \frac{1}{\log \varepsilon^{-1}} \frac{N}{\pi} \left(2\pi \int w^2 \right)^{1/2} \left(1 + \frac{1}{\log \varepsilon^{-1}} \ln \left(\frac{4R}{N} \frac{1+R^N}{1-R^N} \right) \right)^{-1/2}. \tag{15}$$

- Example: $\varepsilon = 0.05, N = 8: \quad A_{l0} \approx 11.86; \quad A_l \approx 14.66$, 25% difference.

Small eigenvalues I

- Linearize equations of motion
- In the limit $A \gg O(1/\log \varepsilon^{-1})$ (when all spikes have same height to leading order)

$$\Lambda(m) = -\sum_{l=0}^{N-1} \left(J_{rr}(R,R,l) + J_{rR}(R,R,l) z^l \right) \tag{16}$$

where $z = \exp(2\pi m i/N)$; $\quad J(r,R,l) = \begin{cases} G(r, Re^{i2\pi l/N}), & \text{if } l \neq 0 \pmod N \\ \frac{1}{2\pi}\log \varepsilon^{-1} + H(r,R), & \text{otherwise} \end{cases}$

– Stability depends only on annulus thickness and N (not on A)

– Most unstable mode is $m = N/2$

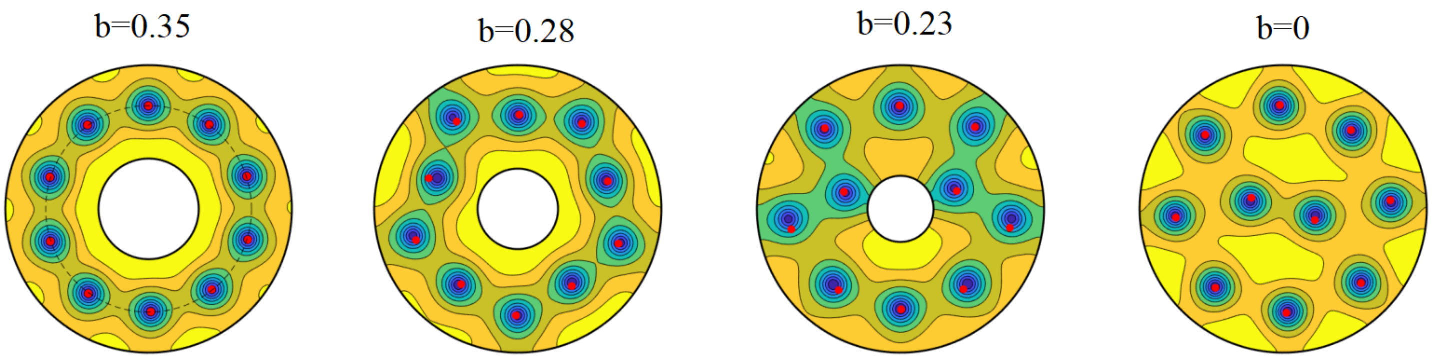
– "Explicit" formula for a disk:

$$2\pi\Lambda(N/2) = -NR^{2N-2}\frac{\left(N-1+R^{2N}\right)+N\left(R^N+R^{-N}\right)}{\left(1-R^{2N}\right)^2} + \frac{1}{8R^2}(N-2)^2 - N$$

– Disk: 8 or less spots are stable, 9 or more are unstable.

– Annulus $b < |x| < 1$: N spikes stable when $b > b_c(N)$:

N	$b_c(N)$
9	0.174
10	***0.293***
11	0.356
12	0.412
13	0.450
14	0.488
15	0.516
16	0.545
17	0.567
18	0.589
19	0.607
20	0.625



Small eigenvalues II

- Don't assume all the heights are the same; assume $A = O(1/\log\varepsilon^{-1})$.

- Linearlized equations are:

$$\frac{\lambda}{S^2} = \Lambda - \frac{1}{\frac{\eta\int w^2}{S^2} - \sum_{l=0}^{N-1} z^l J}\left(\sum_{l=0}^{N-1} J_r z^l\right)^2$$

- Setting $\lambda = 0$ we get a critical threshold:

$$A_s(m) = \frac{N\left(2\pi\int w^2\right)^{1/2}}{\log\varepsilon^{-1}|\Omega|}\left\{1 + \frac{2\pi}{\log\varepsilon^{-1}}\left[\tilde{\Upsilon}(R,R,m) + \frac{\Upsilon_r^2(R,R,m)}{\Lambda(m)}\right]\right\}^{-1/2} \tag{17}$$

$$A_s = \max_m A_s(m) = A_s\left(\lfloor N/2\rfloor\right) \tag{18}$$

- Recall the large eigenvalue threshold:

$$A_l\left(m\right) = \frac{N\left(2\pi\int w^2\right)^{1/2}}{\log\varepsilon^{-1}|\Omega|}\left(1 + \frac{2\pi}{\log\varepsilon^{-1}}\tilde{\Upsilon}(m)\right)^{-1/2}; \tag{19}$$

  – If ring is stable for large A then Λ is negative; this means that $A_s > A_l$.

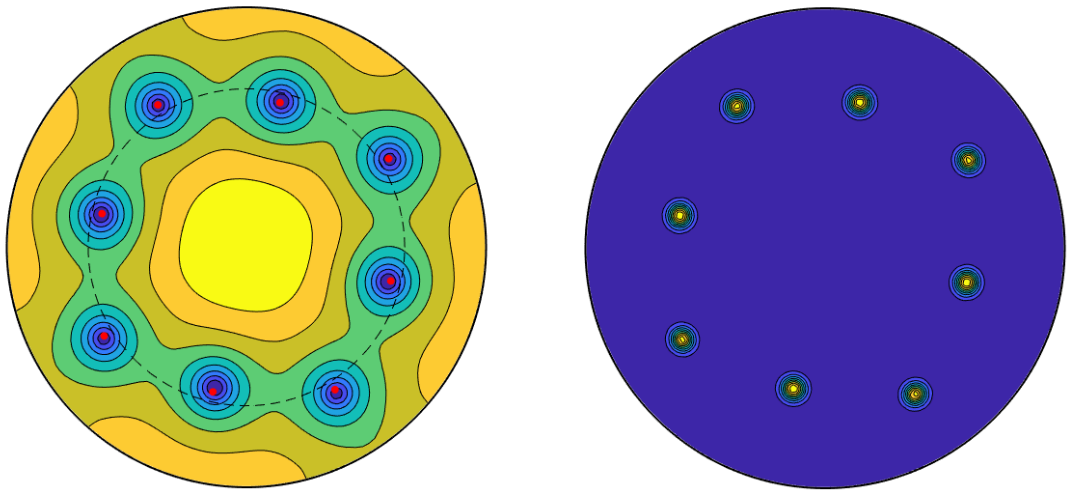
– Conclusion: small eigenvalue instability is triggered before the large eigenvalue instability

– But $\frac{A_s - A_l}{A_l} = O(1/\log \varepsilon^{-1})$. and $A_s \sim A_l$ as $\varepsilon \to 0$.

	$\varepsilon = 0.02$			$\varepsilon = 0.05$		
N	A_s	A_l	$A_{l,0}$	A_s	A_l	$A_{l,0}$
2	2.455	2.183	2.271	3.293	2.818	2.965
3	3.481	3.213	3.406	4.577	4.127	4.448
4	5.033	4.698	4.542	6.814	6.201	5.931
5	6.379	5.962	5.677	8.687	7.910	7.414
6	8.119	7.530	6.813	11.35	10.18	8.897
7	9.912	8.946	7.949	14.20	12.18	10.38
8	52.90	10.59	9.084	N/A	14.66	11.86

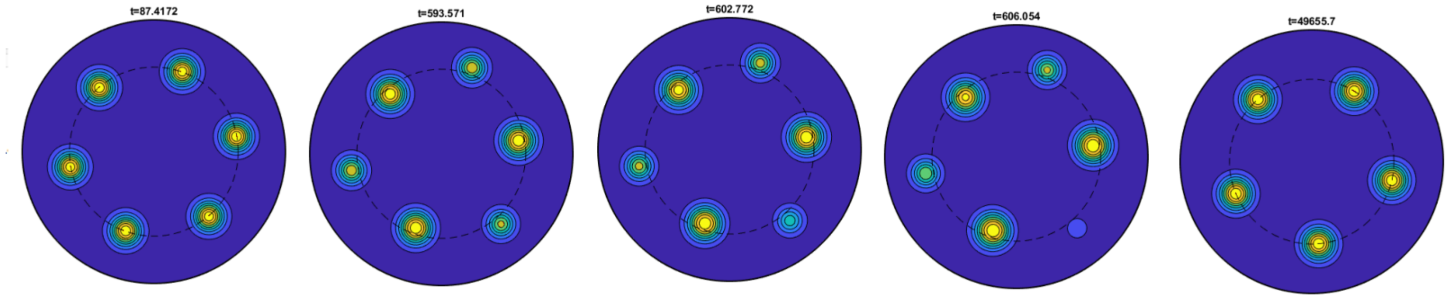
Ring of 8 spikes

- $N = 8: \quad A_s = \frac{39.981}{\log \varepsilon^{-1}} \left\{ 1 - \frac{3.796}{\log \varepsilon^{-1}} \right\}^{-1/2}, \quad A_l = \frac{39.981}{\log \varepsilon^{-1}} \left\{ 1 - \frac{1.035}{\log \varepsilon^{-1}} \right\}^{-1/2}$
  - When $\varepsilon = 0.05$, $1 - \frac{3.796}{\log \varepsilon^{-1}} < 0$ so A_s does not exist; whereas $A_l = 14.66$.
  - When $\varepsilon = 0.02$, $A_s = 52.9$. $A_l = 10.59$ ***But self-replication is observed for*** $A > 20 \implies$ cannot observe a stable 8-ring.
  - E.g. $\varepsilon = 0.02$, $A = 16.7$. 8-Ring deforms into a "square":



Conclusions:

- On a disk:
  - A ring of ≥ 9 spikes is ***always*** unstable due to small eigenvalues
  - A ring of ≤ 8 spikes is stable if $A \gg O(1/\log \varepsilon^{-1})$.
  - The ***dominant*** instability is wrt to small eigenvalues even when $N \leq 8$.



  - The instability is subcritical, ***except*** when $N = 8$.

- On an annulus:
  - The thinner the annulus, the more spikes can be stable
  - The instability is due to small eigenvalues, and is often supercritical.

Under hood I: stability of ring

- ***Restrict*** the motion of kth spike to be along rays $x_k(t) = r_k(t)\exp(2\pi ki/N)$:
  - Simplified problem where all heights are the same:

$$r_k' = -\sum_{l=0}^{N-1} J_r(r_k, r_{k+l}, l) \quad \text{with indices taken} \mod N \tag{20}$$

$$\text{where} \quad J(r,R,l) = \begin{cases} G(r, Re^{i2\pi l/N}), & \text{if } l \neq 0 \pmod N \\ \frac{1}{2\pi}\log\varepsilon^{-1} + H(r,R), & \text{otherwise} \end{cases} \tag{21}$$

- Linearize:

$$r_k = R + \phi_k e^{\lambda t}$$

$$\lambda\phi_k = -\sum_{l=0}^{N-1} \phi_k J_{rr}(R,R,l) + \phi_{k+l} J_{rR}(R,R,l)$$

- "Circulant" anzatz:

$$\phi_k = \phi z^k, \quad z = \exp(2\pi mi/N)$$

$$\Lambda(m) = -\sum_{l=0}^{N-1} \left(J_{rr} + J_{rR} z^l\right)$$

Under hood II: Green's function on annulus

$$G_r(r,R,\theta) = \sum_{n=0}^{\infty} \cos(n\theta)\partial_r g_n(r,R);$$

$$\partial_r g_n = \frac{1}{2\pi r\left(1-b^{2n}\right)}\left(R^n r^n + R^{-n} r^n - b^{2n} R^n r^{-n} - b^{2n} R^{-n} r^{-n}\right), \quad r<R, \quad n \geq 0$$

- Need to compute: $f(1,b^2)$ where $f(\rho,a) = \sum_{n=1}^{\infty} \frac{\rho^n}{1-a^n}$
  - Key insight: ***resummation trick*** ("Ewald summation"):

$$\sum_{n=1}^{\infty} \frac{\rho^n}{1-a^n} = \sum_{n=1}^{\infty} \rho^n \sum_{p=0}^{\infty} a^{np} = \sum_{p=0}^{\infty} \sum_{n=1}^{\infty} (\rho a^p)^n = \sum_{p=0}^{\infty} \frac{\rho a^p}{1-\rho a^p}.$$

  - Similar trick for N-ring interaction:

$$\sum_{l=1}^{N-1} \sum_{n=1}^{\infty} \frac{\rho^n}{1-a^n} \cos\left(\frac{2\pi l}{N} n\right) = \sum_{p=0}^{\infty} \left(N \frac{\rho^N a^{Np}}{1-\rho^N a^{Np}} - \frac{\rho^{p+1}}{1-a^{p+1}}\right)$$

Under hood III: explicit computations for a disk

- Key calculation: $\Lambda(m) = -\sum_{l=0}^{N-1} \left(J_{rr} + J_{rR} z^l \right)$

$$J_r(r, R, \theta) = \frac{1}{2\pi r} \sum_{n=1}^{\infty} \cos(n\theta) \left(R^n r^n + R^{-n} r^n \right)$$

$$T(\rho) := \sum_{l=1}^{N-1} \sum_{n=1}^{\infty} \cos\left(\frac{2\pi n l}{N} \right) e^{2\pi m l i / N} \rho^n, \quad m \neq 0. \tag{22}$$

Then

$$T(\rho) = \frac{N}{1-\rho^N} \left(\frac{\rho^m + \rho^{N-m}}{2} \right) - \frac{\rho}{1-\rho}$$

and

$$T(1) = 1/2; \qquad T(1) = \frac{1-N^2}{12} + \frac{m}{2} (N-m) \,.$$

End result:

$$\Lambda\left(N/2\right) = -N R^{2N-2} \frac{\left(N-1+R^{2N}\right) + N\left(R^N + R^{-N}\right)}{\left(1-R^{2N}\right)^2} + \frac{1}{8R^2}\left(N-2\right)^2 - N$$

- For the annulus, J is evaluated as above, then a sum is evaluated numerically...