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Agent-based model of the effect of globalization on inequality and class mobility

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H I G H L I G H T S

- We propose a model which incorporates the effect of globalization on class mobility.
- The model illustrates a transition wealth mobility versus wealth stratification.
- Increasing connectivity leads to abrupt transitions in inequality and mobility.
- Wealth hotspots appear at small connectivity and dissolve at large connectivity.
- This model recovers both the Kuznets curve as well as the "Great Gatsby" curve.

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A B S T R A C T

We consider a variant of the Bouchaud–Mézard model for wealth distribution in a society which incorporates the interaction radius between the agents, to model the extent of globalization in a society. The wealth distribution depends critically on the extent of this interaction. When interaction is relatively local, a small cluster of individuals emerges which accumulate most of the society's wealth. In this regime, the society is highly stratified with little or no class mobility. As the interaction is increased, the number of wealthy agents decreases, but the overall inequality rises as the freed-up wealth is transferred to the remaining wealthy agents. However when the interaction exceeds a certain critical threshold, the society becomes highly mobile resulting in a much lower economic inequality (low Gini index). This is consistent with the Kuznets upside-down U shaped inequality curve hypothesis.

1. Introduction

Changes in societal structure are driven in large part by the forces of globalization. There is a continuing debate on the effect of globalization on wealth distribution. The relationship is complex and depends on a multitude of factors, including the type of globalization [1] and the level of the country's development [2]. Kuznets [3] postulated a famous hypothesis that the economic inequality generally follows an inverted-U shape as a function of development. Recent studies [4–6] have proposed that the inequality as a function of society's "openness" has a similar shape.

One of the simplest agent-based models of wealth distribution is the kinetic or "gas-collision" model motivated by ideal gas distribution in physics [7,8]. Consider a society of n individuals, each having a certain amount of dollars to start with. At each instant, a "winner" and a "loser" are chosen at random, with the winner receiving one dollar from the loser, provided that the loser has at least a dollar to give. After many such trades, the wealth distribution settles to an exponentially decaying distribution, identical to Boltzmann's distribution for free gases. Variants of this model, where the amount of trade scales with the winner's wealth, lead to algebraically decaying tails (e.g. Pareto distribution) [9,10]. Many related agent models have been proposed, which lead to similar distributions [10–12]; see e.g. [13–15] for a recent surveys.

While kinetic models capture realistic wealth distributions [16,17], these models do not capture the degree of mobility within a society. There is a high correlation between intergenerational mobility and inequality [18–20]—the so-called "Great Gatsby" curve. On the other hand, simulations of trader models such as [8,10,11] show a continual upwards and downwards mobility of individuals, even when the overall Gini index[1] is high. That is, while the overall distribution remains roughly the same with time, each individual's wealth fluctuates, so that the long-time average wealth of each individual is the same.

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[1] The Gini index is the standard measure of inequality in a population. For n individuals with wealth x_j, the Gini index is $\sum_i\sum_j|x_i - x_j|/(2n\sum_j x_j)$.

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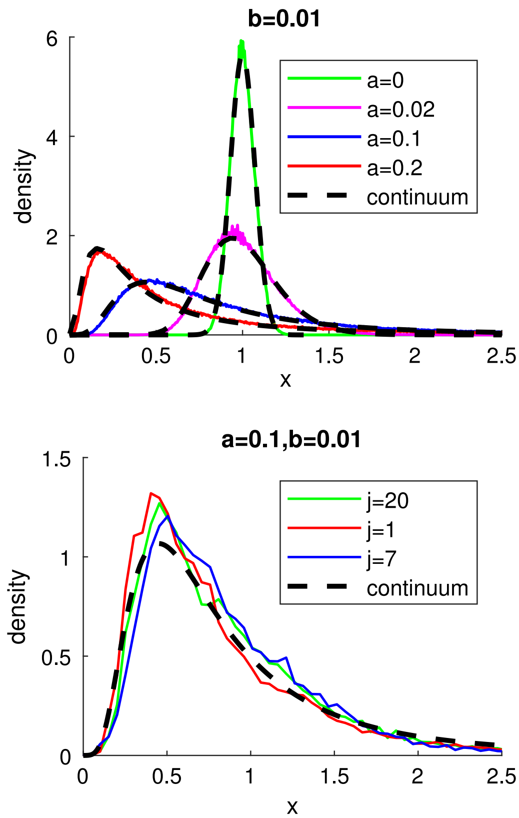

Fig. 1. Wealth distribution for wage earners model with full connectivity. Top: density for several values of a, b as given. Dashed line is the limiting distribution given by (3). In all cases, $n = 100$ with mean $\bar{x} = 1$ (so that the total wealth is 100). Bottom: Distribution of three randomly chosen individual agents *in time* is shown with a solid curve. Stationary distribution (3) of the society as a whole is shown with a dashed curve. It demonstrates that all individuals have identical distribution in time, which is also identical to the stationary distribution of the society as a whole. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Here, we use a variant of the Bouchaud–Mézard model [11] to capture the transition from high class mobility to a highly stratified society using an agent-based framework. This is done by incorporating the notion of spatial distance between the agents. Instead of pairwise interactions as in the gas-collision model of wealth, or all-to-all interactions as in the original Bouchaud–Mézard model, we consider interactions of neighbours within a certain radius R. By considering the mean-field limit, we examine how the inequality level depends on R. A key finding of this model is that for sufficiently low R and when the return on investments is sufficiently high, the society is highly stratified with little or no class mobility and high inequality. As the connectivity is increased past a certain critical threshold, an instability is triggered resulting in a sudden drop of inequality and high class mobility. This effect is similar to the upside-down U Kuznets curve.

2. All-to-all connectedness

Before presenting a model with partial connectivity, let us start by considering a simple model of "wage earning", which is a variant of the Bouchaud–Mézard model. Choose an agent j at random and increase its wealth x_j by w_j where

$$w_j = ax_j + b\bar{x}. \tag{1}$$

Here, $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$ is the average wealth of the society and parameters a, b represent wealth generation through investments and fixed income, respectively. We assume that both a, b are positive. Since we are interested in the relative wealth between the agents, after each step, we rescale each agent's wealth so that the overall total wealth is conserved. Since the total amount of money T is increased by w_j, this means rescaling the wealth of every agent by the amount $T/(T + w_j)$. This can be thought as an inflationary decrease of the value of money. Repeating this process multiple times yields a stationary distribution which depends on a, b.

The model (1) is very similar to the Bouchaud–Mézard model [11]. The difference is where the randomness comes in. In the Bouchaud–Mézard model, the *parameters* a, b are stochastic, whereas here, we pick the *agents* at random at each time step, in analogy to [8].

In Appendix A we show that in the limit $a, b \ll O(1)$, the resulting density distribution $u(x)$ is approximated by an ODE

$$\frac{1}{2}\left((ax + b)^2 u\right)'' + (b(x - 1)u)' = 0 \tag{2}$$

where (without loss of generality) we scaled the mean to be one: $\bar{x} = 1$. This ODE admits an *exact* solution

$$u(x) = C\left(x + \frac{b}{a}\right)^{-\left(2+2\frac{b}{a^2}\right)} \exp\left(-\frac{2b\,(a+b)}{a^3}\,\frac{1}{\left(x + \frac{b}{a}\right)}\right), \tag{3}$$

where the normalization constant C is chosen so that $\int_0^\infty u = 1$.

There are two distinguished limits of (2): the low investment and high investment regimes. The low investment regime is when $a = O(b)$ or $a \ll O(b)$. In this case, the distribution asymptotes to a Gaussian with mean $\bar{x} = 1$ and standard deviation $\sigma = (a + b)\sqrt{1/(2b)}$; see the trend indicated by the purple and green curves in Fig. 1(top). The wealth distribution concentrates around the mean, yielding a relatively small Gini index (with $Gini \approx 0.56(a + b)\sqrt{1/(2b)}$)

On the other hand, the "high investment" regime occurs when $a \gg O(b)$; cf. the blue and red curves in Fig. 1(top). In this regime, distribution (3) reduces to the well-known Bouchaud–Mézard distribution [8],

$$u(x) = Cx^{-p} \exp\left(-\left(p - 2\right)/x\right), \quad p = 2 + 2\frac{b}{a^2}. \tag{4}$$

Regardless of the values of a, b this model exhibits *class mobility*: rich agents will eventually become poor and vice-versa, if the simulation goes on for long enough. In other words, every agent has the same distribution of wealth in time, which is also identical to the stationary distribution of the society as a whole. This is illustrated in Fig. 1(bottom). In many real societies however, the wealth often tends to be "sticky", with inter-generational wealth transfer correlating strongly with high inequality.

3. Near-neighbour interactions

To account for stratification of wealth, we now introduce near-neighbour interactions. The wealth accumulation still follows (1) except that the rescaling is restricted to an R-neighbourhood of x_j. This can be thought of as a local devaluation. In other words, the updated values are:

$$\hat{x}_k = \begin{cases} (x_j + w_j)\dfrac{T_j}{T_j + w_j}, & \text{if } k = j \\ x_k\dfrac{T_j}{T_j + w_j}, & \text{if } k \in N_j \setminus \{j\} \\ x_k, & \text{otherwise} \end{cases} \tag{5}$$

where N_j is the R-neighbourhood of x_j and T_j is the total wealth of the *neighbourhood*:

$$N_j = \{j : j \in \mathbb{Z},\ |j - k| \le R\}\,; \quad T_j = \sum_{i \in N_j} x_i. \tag{6}$$

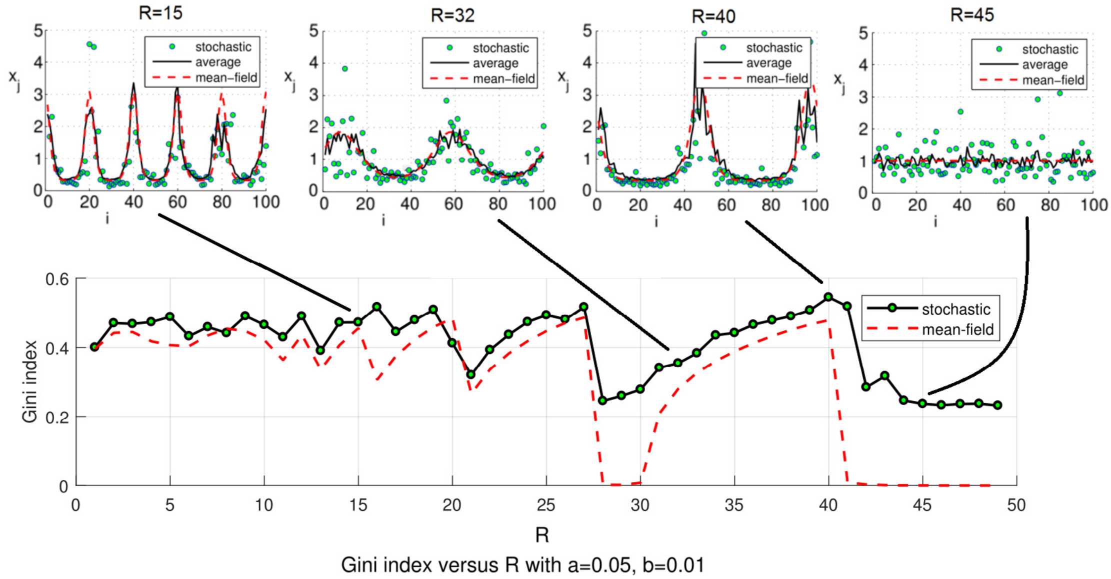

Fig. 2. Top: Comparison of stochastic model (5) versus mean-field limit model (7) for several values of R with $a = 0.05$, $b = 0.01$. Green dots give a snapshot of a simulation after 10^6 steps. The black line shows the average wealth over 10^6 simulations of the stochastic model whereas the dashed red line is the steady state of the corresponding mean-field limit. Bottom: Gini index as a function of the connectivity radius R with $a = 0.05$, $b = 0.01$. Dots are generated by simulating (5), starting with random initial conditions; dashed line is the simulation of the mean-field model (7). Note multiple abrupt changes including a window of mobility $28 \le R \le 30$.

We impose periodic boundary conditions by placing the agents on a ring of length n so that the indices are wrapped around mod n. The case of full connectivity is just a special case of this rule corresponding to $R = (n-1)/2$ when n is odd.

Fig. 2(top) shows the simulation of (5) for several values of trade distance R, with $a = 0.05$, $b = 0.01$. When R is small, several "spikes" of wealthy agents emerge. As R is increased, there is wealth consolidation, with fewer and fewer spikes remaining. Eventually, the spikes disappear entirely when the neighbourhood size is increased past $R_{\max} = 42$, about 85% of the domain. This corresponds to *sharp transition* from a highly stratified and unequal society to a much more egalitarian society with high class mobility.

This model recovers naturally the "Great Gatsby" correlation between the inequality and intergenerational mobility. To illustrate this, consider the Gini index. The Gini index of zero corresponds to perfect equality while Gini index of one is the extreme "pauper" society where a single person has all of the society's wealth. Fig. 2(bottom) shows the Gini index as a function of R, averaged over 10^6 simulations. The areas of high Gini index coincide precisely with a stratified society, and abrupt transitions from stratified to mobile society are accompanied by a corresponding sharp drop in the Gini index.

To understand the origin of this transition, consider the mean-field limit of this model as derived in Appendix B. We show that the long-time average of agent x_k satisfies a *deterministic* ODE system

$$\frac{dx_k}{dt} = w_k - x_k \sum_{j \in N_k} \frac{w_j}{T_j}, \tag{7}$$

where w_j, T_j and N_k are given in (1), (6). This system admits a *constant* steady state $x_k = \bar{x}$, which represents a society with high degree of mobility, in other words, this is the state where each agent's long-time average wealth is the same. We show in Appendix B that this state is stable when $a < a_c$ where a_c is given by

$$a_c = b\left(-1 - \frac{1}{\min_{m=1,\ldots,n} \frac{\sin\left(\frac{\pi m}{n}(2R+1)\right)}{(2R+1)\sin\left(\frac{\pi m}{n}\right)}}\right). \tag{8}$$

This curve is shown as dashed red line in Fig. 3. As a is increased past a_c, the population suddenly transitions from highly mobile to highly stratified, and this is accompanied by a sudden transition from low to high inequality (low to high Gini index). The resulting instability represents spontaneous breaking of spatial translation symmetry along the ring where the agents are placed, and represents spontaneous inequality enhancement and decrease of wealth mobility, as illustrated in the top row of Fig. 2.

A similar bifurcation occurs with respect to R, provided that $a > a_{\min}$, where $a_{\min} = 3.6033 \cdot b = \min_R a_c$ is shown by the dashed horizontal line in Fig. 3. In fact, multiple transitions from stratified to mobile society and back can occur as R is gradually increased, as long as $a/b < 6$. This is what we see in Fig. 2(bottom), which is in fact a horizontal slice of Fig. 3 at $a/b = 5$. The windows of mobility disappear when $a/b > 6$, but the Gini index still undergoes multiple abrupt jumps as R is increased, until R is increased past $R_{\max}$ (shown with a green dashed curve in Fig. 3), at which point the society becomes mobile and does not return to a stratified state.

More generally, stratification first occurs when R is sufficiently small while the return on investment a is sufficiently large ($a > a_{\min}$). For a relatively small R (i.e. local economy or little globalization), stratification sets in suddenly when a is increased past $a_{\min}$. In the stratified regime a number of "hot-spots" of high wealth forms, see Fig. 2(top). As R is increased, the number of hot-spots diminishes, which leads to a very complex relationship between the Gini index and R, refer to Fig. 2. The inequality is the highest just before a reduction in the number of hot-spots. In the aftermath of a hot-spot disappearance there is a sharp drop in the Gini index as the wealth is freed-up. However most of the freed-up wealth is absorbed by the remaining hot-spots, leading to a consequent increase in the Gini index as R continues to be increased. This process of redistribution is repeated many times over, until only two hot-spots are left. Once R is increased past $R_{\max}$, the last two spots suddenly disappear, and the society abruptly transitions into a highly mobile regime with low inequality.

In the limit $R \gg 1$, $n \gg 1$, the transition boundaries between m and $m+1$ spots (vertical lines in Fig. 3) correspond to the jump in the most unstable mode, and satisfy $\sin(\pi m/x)/(m\pi/x) = \sin(\pi(m+1)/x)/(\pi(m+1)/x)$ where $x = n/(2R+1)$ is the

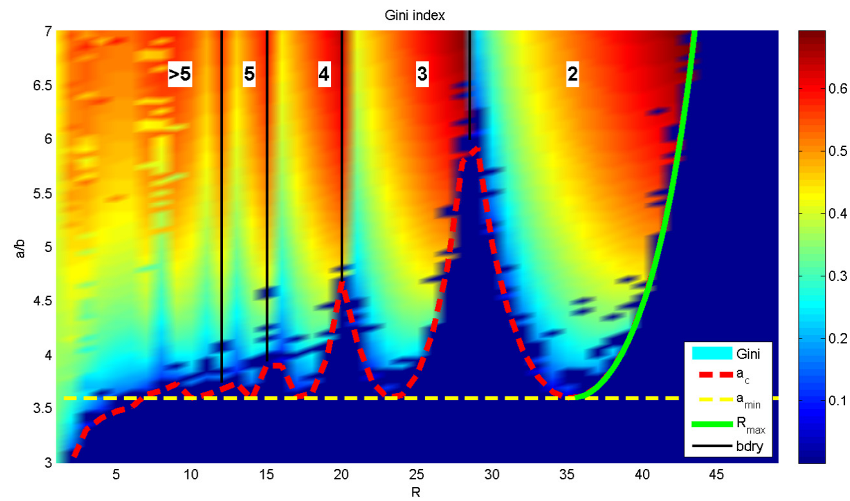

Fig. 3. Gini index as a function of the connectivity radius R and investment to wage ratio a/b. The figure is generated by simulating the continuum model (7) for each choice of $(a/b, R)$. The black vertical lines (denoted by 'bdry' in the legend) are the boundaries separating the regions with different number of hot-spots. Digits at the top represent the number of hot-spots that arise. Initial conditions are $x_j = 1$ for $j \neq 50$ and $x_{50} = 2$.

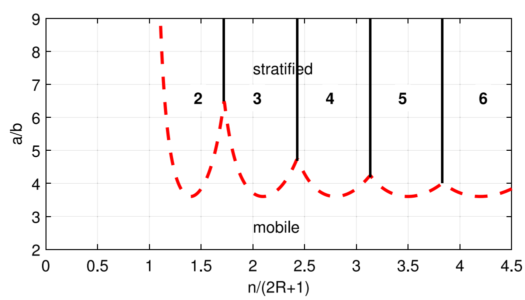

Fig. 4. Boundary between mobile and stratified society. Dashed line is a_c given by (8). Society is mobile below the dashed line. Above the dashed line, hot-spots of wealth form. Vertical lines denote the boundaries between different number of hot-spots, as indicated by the digit.

first root to the left of $x = m$. The first few such values are: $x_2 = 1.7228$, $x_3 = 2.4297$, and more generally, $x_m \sim 0.6991m$ as $m \to \infty$. (The constant $z = 0.6991$ is the biggest root $\tan(\pi/z) = \pi/z$.) Note that in the regime where $n/(2R + 1) \gg 1$ (with $n, R \gg 1$), the number of wealth hot-spots that is roughly proportional to $n/(2R + 1)$. This is illustrated in Fig. 4.

Finally, we mention that in the limit $R \gg 1, n \gg 1$ we can take the continuum limit of the ODE system (7). We discretize $x_k \sim u(z)$, $z = k/n$; $z \in [0, 1]$. Then the steady state satisfies the integral equation

$$\int_{-r}^{+r} \frac{au(z+s)+b}{\int_{-r}^{r} u(z+s+y)dy} ds = \frac{au(z)+b}{u(z)}, \quad r = R/n. \tag{9}$$

It is an open question to obtain any results about this system (beyond the linear stability analysis).

The rise of inequality followed by a sudden drop as the connectivity is increased is qualitatively similar to the upside-down U curve postulated by Kuznets [3]. The difference here is in the abruptness of the transition from mobility to stratification, as well as the possibility of multiple such transitions. Our model provides for a nice demonstration of highly nonlinear response of inequality to the increased trade and globalization. An extreme example is the rise of robber barons during the guilded age in the 1870s (which was precipitated by increased trade due to the development of railroads) and the corresponding dramatic rise in inequality. A more recent example is the consolidation in the retail sector with the rise of Amazon and increased use of online shopping [21]. From 2006 to 2016, Amazon's share of the market rose from 4% to 54%, while the second closest competitor (Walmart) dropped from 51% to 32%. The Gini index among the top nine companies that dominate the retail sector rose from 0.56 to 0.81 in the same time frame. Certainly increased connectivity due to internet played a key role in the upheaval of the industry.

The phenomenon of stratification with increased connectivity, followed by transition to mobility when connectivity is sufficiently global is not limited to near-neighbour interactions. Instead of "geographical" connections where the local devaluation is within all neighbours a distance R away, one can consider more general interactions. An example of this is shown in Fig. 5, where we used a random undirected graph (Erdös–Rényi graph) instead of a geographical neighbourhood graph. Such a graph is constructed by connecting any two nodes with probability $p \in (0, 1)$. The figure shows the Gini index versus the average degree. Note the upside-down U shape, consistent with Kuznets hypothesis. Also note a sharp drop in Gini index for sufficiently high connectivity, similar to the R-neighbourhood model. Unlike the R-neighbourhood graph where each node has the same degree, the nodes in Erdös–Rényi graph have Poisson degree distributions. In either model, the transition to low Gini index only happens when the connectivity is very high, close to all-to-all.

4. Comparison with Bouchaud–Mézard model

Let us contrast our model (5) with the Bouchaud–Mézard model with local interactions [11]. The general model considered there is a set of stochastic ODE's

$$dx_k = (\sqrt{dt}\sigma\xi + dt\, m)x_k + \sum_j dtJ_{kj}x_j - \sum_j dtJ_{jk}x_i \tag{10}$$

where $\sigma\xi$ is a Gaussian random variable of standard deviation σ and mean zero. We remark that in the case of all-to-all connectedness $J_{jk} = J$, the authors show that the density distribution is given by the well-known formula (4), except that $p = 2 + \frac{J}{\sigma^2}$. In effect, the noise level σ plays the role of a whereas the trade strength J plays the role of b. In this sense, the Bouchaud–Mézard model with all-to-all interactions is a special case of model (1), in the limit $a^2 = O(b)$.

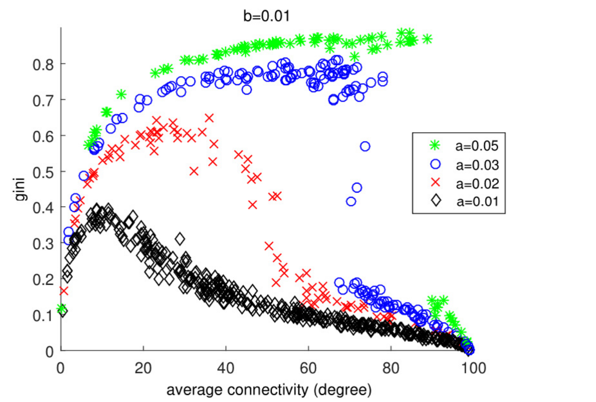

Fig. 5. Gini index for the model (5) but with random (Erdös–Rényi graph) connectivity. Here, $b = 0.01$, $n = 100$ and a is as indicated. For higher values of a there is a sudden transition to low Gini index as the connectivity is increased. For each value of a, 1000 simulations are shown for randomly chosen values of edge probability p.

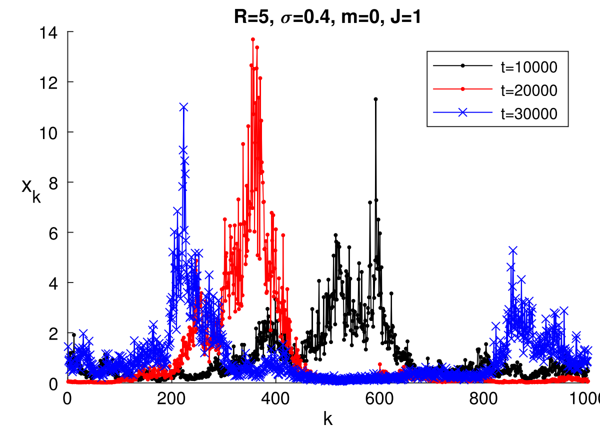

Fig. 6. Simulation of Bouchaud–Mézard model with near-neighbour interactions (12). Parameters as shown in the title. Spatially-correlated patterns of wealth concentrations form, but shift over long time-scales.

In [11] the authors consider a special case

$$J_{kj} = \begin{cases} \dfrac{J}{2R+1}, & \text{if } |k-j| \le R \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

corresponding to having local interactions within an R-neighbourhood, so that (10) becomes

$$dx_k = (\sqrt{dt}\sigma\xi + dt\ m)x_k + dtJ\left(\frac{\sum_{j=k-R}^{k+R} x_j}{2R+1} - x_k\right). \tag{12}$$

The authors show that for sufficiently small J/σ^2 and with small R, the model exhibits *wealth condensation* where very few agents acquire most of the wealth, with a correspondingly very high Gini index (e.g. $gini = 0.8$). The tell-tale sign of condensation is a degenerate fat power-law tail distribution: $u(x) \sim Cx^{-p}$ with $p \le 2$ for large x. However, this is not the same as *wealth stratification:* even in the condensation regime, the long-time average of agent's wealth remains the same. To see this, consider the limit $\sigma \to 0$. Changing variables $x_k = e^{mt}y_k$ yields the ODE system

$$y_k' = J\left(\frac{\sum_{j=k-R}^{k+R} y_j}{2R+1} - y_k\right)$$

admitting a constant steady state $y_k = y$. Linearizing around this constant state yields the eigenvalues

$$\lambda = J\left(\frac{\sin\left(\frac{2\pi m}{n}(2R+1)\right)}{(2R+1)\sin\left(\frac{2\pi m}{n}\right)} - 1\right), \quad m = 1, \dots, n,$$

which are all negative. As a result, there is no stratification expected in this model as $\sigma \to 0$. This is confirmed with direct numerical simulations. Nonetheless, for sufficiently large σ and small R, relatively long transient populations are observed as illustrated in Fig. 6

5. Conclusion

In conclusion, we have proposed an agent-based model of wealth mobility and inequality, based on economic growth incorporating local devaluation within a certain connectivity radius. This model exhibits two fundamental regimes: wealth mobility versus wealth stratification. In the high mobility regime, each individual agent's wealth fluctuates with time, but all agents have the same wealth distribution with respect to time (and the same long-time average). By contrast, in the high stratification regime, the long-time average wealth of different agents is *not* the same, leading to caste-type societal structure. In this regime, few rich agents accrue most of the societal wealth, and *they remain rich* for the entire simulation, while poor agents remain poor. Increasing connectivity leads to (possibly multiple) abrupt transitions in inequality and mobility, finally resulting in a highly mobile society for sufficiently large connectivity. This model recovers both the Kuznets curve as well as the "Great Gatsby" curve. It also underscores the complex nonlinear relationship between the inequality and connectivity in the society.

Appendix A. Derivation of distribution for all-to-all connectedness

We now derive the distribution of wealth $u(x)$ of each agent in the all-to-all coupling case. Let $K(x, y)$ denote the probability that an agent starting with wealth y before the update step, will have wealth x after the update step. The distribution $u(x)$ then satisfies

$$u(x) = \int K(x, y)u(y)dy.$$

Here and below, we assume that $u(x)$ is zero for negative x. By scaling, we assume without loss of generality that $\bar{x} = 1$ so that $T = \sum x_j = n$ and $w_k = ax_k + b$. Therefore the update rule is

$$x_k \mapsto \begin{cases} \dfrac{n\,(x_k + w_k)}{n + w_k}, & \text{with probability } \dfrac{1}{n}; \\ \dfrac{n\,x_k}{n + w_j}, & \text{for each } j \ne k \text{ with probability } \dfrac{1}{n}. \end{cases}$$

Then $K(x, y)$ can be estimated as

$$K(x, y) = \frac{1}{n}\,\delta\left(x - \frac{n\,(y + (ay + b))}{n + (ay + b)}\right) + \frac{n-1}{n}\int \delta\left(x - \frac{n\,y}{n + (az + b)}\right) u(z)\,dz \tag{13}$$

which yields the master equation

$$u(x) = \frac{1}{n}\int \delta\left(x - \frac{n\,(y + (ay + b))}{n + (ay + b)}\right) u(y)\,dy + \frac{n-1}{n}\iint \delta\left(x - \frac{n\,y}{n + (az + b)}\right) u(z)\,dz\,u(y)\,dy. \tag{14}$$

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The first integral evaluates to

$$I_1 = \frac{1}{n}\int \delta\left(x - \frac{n\,(y+(ay+b))}{n+(ay+b)}\right) u(y)\,dy$$
$$= \frac{1}{n}\frac{1}{1+a}u\left(\frac{x-b}{1+a}\right) + O\left(n^{-2}\right).$$

To evaluate the second integral, switch the order of integration. This yields,

$$I_2 = \frac{n-1}{n}\iint \delta\left(x - \frac{n\,y}{n+(az+b)}\right) u(z)\,dz\,u(y)\,dy$$
$$= \frac{n-1}{n}\int\left(\int \delta\left(x - \frac{n\,y}{n+(az+b)}\right)u(y)dy\right)u(z)dz$$
$$= \frac{n-1}{n}\int u\left(x + \frac{x\,(az+b)}{n}\right)\left(1+\frac{az+b}{n}\right)u(z)dz.$$

A two-order expansion in n yields

$$I_2 = u(x) + \frac{1}{n}\left\{u'(x)\,(a+b)\,x + u(x)\,(a+b-1)\right\} + O\left(n^{-2}\right)$$

where we used the facts that $\int u(z) = \int zu(z) = 1$. Thus, to leading order in n, the master equation (14) becomes

$$0 = \frac{1}{1+a}u\left(\frac{x-b}{1+a}\right) + u'(x)\,(a+b)\,x + u(x)\,(a+b-1)\,. \tag{15}$$

Expanding up to $O((a,b)^2)$ yields a second-order ODE,

$$0 = \frac{1}{2}(ax+b)^2u''(x)$$
$$+ \left(2a(ax+b) + b\,(x-1)\right)u'(x) + \left(a^2+b\right)u(x).$$

This ODE is exact and can be rewritten as

$$0 = \frac{1}{2}\left((ax+b)^2u(x)\right)'' + \left(b\,(x-1)\,u(x)\right)'. \tag{16}$$

Integrating, assuming decay for large x, yields a first order ODE

$$0 = \frac{1}{2}\left((ax+b)^2u(x)\right)' + \left(b\,(x-1)\,u(x)\right)$$

whose solution is given by

$$u(x) = C\left(x+\frac{b}{a}\right)^{-\left(2+2\frac{b}{a^2}\right)}\exp\left(-\frac{2b\,(a+b)}{a^3}\frac{1}{\left(x+\frac{b}{a}\right)}\right); \tag{17}$$

where C is chosen so that $\int_0^\infty u = 1$.

We remark that the ODE (16) can be viewed as a stationary state of the Fokker–Planck equation $u_t + (A(x)u)_x = \left(\frac{1}{2}B^2(x)u\right)_{xx}$ for a random process with a space-dependent diffusion of the form $B(x) = ax+b$ as well as drift of the form $A(x) = b(1-x)$. This point of view was used in [11] to derive the original Bouchaud–Mézard distribution, as well more general class of distributions [13,22], including distributions that fit the empirically observed distributions in the US [22].

Appendix B. Continuum limit and stability threshold for R-neighbour model

Here, we derive the continuum limit (7) and its stability thresholds (8). Suppose that all agents are updated every dt time step on average. Relabel $w_k \to dt\,w_k$ so as to obtain a scale-free model in the limit $dt \to 0$. Then the rule (5) becomes

$$x_k(t+dt) = x_k + \left(\sum_{j\in N_k\setminus\{k\}}\left(x_k\frac{T_j}{T_j+dt\,w_j} - x_k\right)\right)$$
$$+ \left((x_k + dt\,w_k)\frac{T_k}{T_k+dt\,w_k} - x_k\right).$$

Expanding in Taylor series yields

$$\frac{dx_k}{dt} = w_k - x_k\sum_{j\in N_k}\frac{w_j}{T_j}. \tag{18}$$

which is precisely the ODE model (7). A trivial steady state of (18) is the constant state $x_k = \bar{x}$. We linearize around $\bar{x}$,

$$x_k = \bar{x} + \phi_k e^{\lambda t},\quad \phi_k \ll 1$$

to obtain the following eigenvalue problem,

$$\lambda\phi_k = a\phi_k - \phi_k\,(a+b) - \frac{1}{2R+1}\sum_{j\in N_k}a\phi_j \tag{19}$$
$$+ \frac{a+b}{(2R+1)^2}\sum_{j\in N_k}\sum_{l\in N_j}\phi_l. \tag{20}$$

This problem admits the following self-consistent ansatz:

$$\phi_k = z^k,\ \text{where } z = \exp(2\pi im/n), m = 1,\dots,n. \tag{21}$$

Define the *Dirichlet kernel*,

$$F(\theta) := \frac{1}{2R+1}\sum_{l=-R}^{R}\exp(i\theta l) = \frac{1}{2R+1}\frac{\sin\left((R+1/2)\,\theta\right)}{\sin(\theta/2)}. \tag{22}$$

Let $\theta = 2\pi m/n$. We have

$$\frac{1}{2R+1}\sum_{j\in N_k}\phi_j = z^kF(\theta),$$

and

$$\sum_{j\in N_k}\sum_{l\in N_j}\phi_l = z^k\sum_{j=-R}^{R}z^j\sum_{l=-R}^{R}z^l = z^kF^2(\theta)(2R+1)^2$$

so that upon substituting (21) into (19) we obtain

$$\lambda = -b - aF(\theta) + (a+b)\,F^2(\theta),\ \theta = \frac{2\pi m}{n},\ m = 1,\dots,n.$$

Note that when $a = 0$, we have $\lambda = b\left(-1+F^2\right) \leqslant 0$ since $F^2 \in [0,1]$. On the other hand, when a is large enough, we have $\lambda \sim a(F^2-F)$. From the explicit formula (22) it is easy to show that there is an $1 \leqslant m \leqslant n$ such that $F(\theta) < 0$, and for such θ, $\lambda > 0$ for sufficiently large a. So the constant state $x_k = \bar{x}$ is stable for sufficiently small a and is unstable for sufficiently large a. Setting $\lambda = 0$ and solving the resulting expression for a, we obtain the stability threshold $a_c(\theta) = b\left(-1-\frac{1}{F(\theta)}\right)$. Taking the minimum over all admissible θ yields the expression (8).

References

[1] D. Asteriou, S. Dimelis, A. Moudatsou, Globalization and income inequality: A panel data econometric approach for the EU27 countries, Econ. Model. 36 (2014) 592–599.
[2] H. Kratou, M. Goaied, How can globalization affect income distribution? evidence from developing countries, Int. Trade J. 30 (2) (2016) 132–158.
[3] S. Kuznets, Economic growth and income inequality, Amer. Econ. Rev. (1955) 1–28.
[4] J.-e. Lee, Inequality in the globalizing Asia, Appl. Econ. 42 (23) (2010) 2975–2984.
[5] A. Jalil, Modeling income inequality and openness in the framework of Kuznets curve: New evidence from China, Econ. Model. 29 (2) (2012) 309–315.
[6] W. Cheng, Y. Wu, Understanding the Kuznets process: An empirical investigation of income inequality in China: 1978–2011, Soc. Indicators Res. (2016) 1–20.
[7] S. Ispolatov, P.L. Krapivsky, S. Redner, Wealth distributions in asset exchange models, Eur. Phys. J. B 2 (2) (1998) 267–276.
[8] A. Dragulescu, V.M. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (4) (2000) 723–729.
[9] A. Banerjee, V.M. Yakovenko, Universal patterns of inequality, New J. Phys. 12 (7) (2010) 075032.

[10] A. Chakraborti, B.K. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (1) (2000) 167–170.
[11] J.-P. Bouchaud, M. Mézard, Wealth condensation in a simple model of economy, Physica A 282 (3) (2000) 536–545.
[12] F. Slanina, Inelastically scattering particles and wealth distribution in an open economy, Phys. Rev. E 69 (4) (2004) 046102.
[13] V.M. Yakovenko, J.B. Rosser Jr., Colloquium: statistical mechanics of money, wealth, and income, Rev. Modern Phys. 81 (4) (2009) 1703.
[14] B.K. Chakrabarti, A. Chakraborti, S.R. Chakravarty, A. Chatterjee, Econophysics of Income and Wealth Distributions, Cambridge University Press, 2013.
[15] E. Samanidou, E. Zschischang, D. Stauffer, T. Lux, Agent-based models of financial markets, Rep. Progr. Phys. 70 (3) (2007) 409.
[16] A. Drăgulescu, V.M. Yakovenko, Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States, Physica A 299 (1) (2001) 213–221.
[17] Y. Tao, X. Wu, T. Zhou, W. Yan, Y. Huang, H. Yu, B. Mondal, V.M. Yakovenko, Universal exponential structure of income inequality: evidence from 60 countries, 2016. ArXiv Preprint ArXiv:1612.01624.
[18] M. Corak, Income inequality, equality of opportunity, and intergenerational mobility, J. Econ. Perspect. 27 (3) (2013) 79–102.
[19] A. Björklund, M. Jäntti, Intergenerational income mobility and the role of family background, in: Oxford Handbook of Economic Inequality, Oxford University Press, Oxford, 2009, pp. 491–521.
[20] D. Andrews, A. Leigh, More inequality, less social mobility, Appl. Econ. Lett. 16 (15) (2009) 1489–1492.
[21] J. Desjardins, The extraordinary size of amazon in one chart, 2017. http://Www.Visualcapitalist.Com/Extraordinary-Size-Amazon-One-Chart/.
[22] A.C. Silva, V.M. Yakovenko, Temporal evolution of the thermal and superthermal income classes in the usa during 1983–2001, Europhys. Lett. 69 (2) (2004) 304.