Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx
pp. **X–XX**

OSCILLATIONS OF MANY INTERFACES IN THE NEAR-SHADOW REGIME OF TWO-COMPONENT REACTION-DIFFUSION SYSTEMS

Shuangquan Xie

Dalhousie University
Halifax, NS, Canada

Theodore Kolokolnikov*

Dalhousie University
Halifax, NS, Canada

(Communicated by the associate editor name)

Abstract. We consider the general class of two-component reaction-diffusion systems on a finite domain that admit interface solutions in one of the components, and we study the dynamics of n interfaces in one dimension. In the limit where the second component has large diffusion, we fully characterize the possible behaviour of n interfaces. We show that after the transients die out, the motion of n interfaces is described by the motion of a *single* interface on the domain that is $1/n$ the size of the original domain. Depending on parameter regime and initial conditions, one of the following three outcomes results: (1) some interfaces collide; (2) all n interfaces reach a symmetric steady state; (3) all n interfaces oscillate indefinitely. In the latter case, the oscillations are described by a simple harmonic motion with even-numbered interfaces oscillating in phase while odd-numbered interfaces are oscillating in anti-phase. This extends a recent work by [McKay, Kolokolnikov, Muir, DCDS B(17), 2012] from two to any number of interfaces.

1. **Introduction.**

Many models in nature involve two-component reaction-diffusion system of the general form

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(u,w) \\ \tau w_t = D w_{xx} + g(u,w) \end{cases} \tag{1}$$

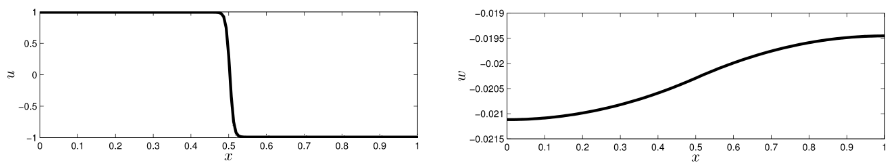
Examples include models of regulatory gene networks [1], wave propagation in excitable media [2], chemical reactions [3, 4, 5, 6], gas discharge dynamics [7, 8], population dynamics [9, 10] and vegetation in arrid landscapes [11]. In the regime $\varepsilon \ll D$ and under certain general conditions on the nonlinearities f, g, this system admits a solution which consists of areas of nearly constant u which are connected by n back-to-back interfaces, where u transitions from one constant value to another. An example of such a solution in shown in Figure 1.

2010 *Mathematics Subject Classification.* Primary: 35K57, 35B36; Secondary: 35B25.

Key words and phrases. Pattern formation, interface oscillation, reaction-diffusion systems.

T.K. is supported by NSERC Discovery Grant No. RGPIN-33798 and Accelerator Supplement Grant No. RGPAS/461907.

* Corresponding author: Theodore Kolokolnikov.



(a) (b)

FIGURE 1. Steady state solution to the model (3) with $x \in [0,1]$, $\beta = 0, \varepsilon = 0.01, D = 150$

It has been known for quite some time [12] that such interfaces can undergo a transition from a stationary state to an oscillatory motion as the paramer τ is increased past some threshold τ_{hopf}; this instability is triggered via a Hopf bifurcation. This was first reported in [12] for a system (1) with piecewise-linear nonlinearities f and g. Since then, similar oscillatory behaviour was reported and analysed in many other reaction diffusion systems in one and higher dimensions, see for example [13, 14, 1, 2, 15, 7, 16, 17, 18, 19, 20, 21, 22, 8]. In a recent paper [23], the oscillatory motion was fully characterized for solutions consisting of one or two interfaces in the regime where D is assumed large. This assumption allowed for a detailed analysis using the method of multiple scales even for values of τ well above τ_{hopf}. The goal of this paper is to extend this analysis from two interfaces to many interfaces. As in the paper [23], we make the following assumptions:

$$D \gg 1, \quad \varepsilon \ll 1, \quad \tau = \tau_0 \frac{D}{\varepsilon}, \quad \text{with } \tau_0 = O(1). \tag{2}$$

Under these assumptions, it was found in [23] that the dynamics of the solution that consists of a single interface is described by a weakly-perturbed harmonic oscillator, and that the envelope of the oscillations can be computed using the method of multiple scales. To illustrate the theory, consider the following system of the general form (1),

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + 2(u - u^3) + w \\ \tau_0 \frac{D}{\varepsilon} w_t = D w_{xx} - u + \beta \end{cases}. \tag{3}$$

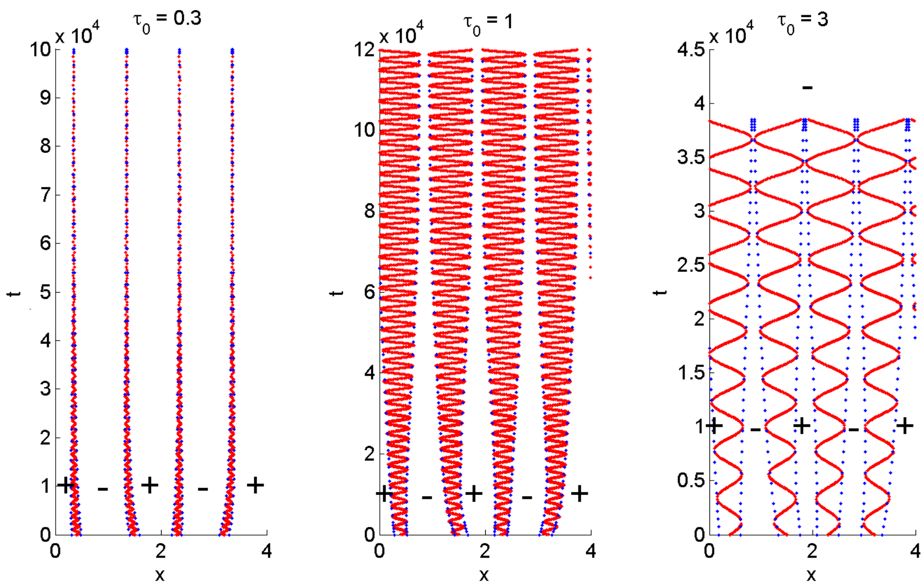
As was shown in [23], this system has a solution in the form of a single interface on a unit interval $x \in [0,1]$ with $u(x,t) \sim \tanh((\xi(t) - x)/\varepsilon)$; $w \sim 0$, where the interface location ξ oscillates according to the formula $\xi(t) = (1+\beta)/2 + A\left(\varepsilon D^{-1} t\right) \cos(\sqrt{3/\tau_0}\varepsilon D^{-1/2} t + \phi_0)$ where the envelope A satisfies

$$A'(\varepsilon D^{-1} t) = \left(\frac{1}{4}(1 - 3\beta^2) - \frac{1}{8\tau_0}\right) A - \frac{3}{4} A^3. \tag{4}$$

Our goal in the present paper is to extend this computation to multiple interfaces. Our main conclusion is succinctly summarized as follows.

Main Result. *After the transients die out, the dynamics of n interfaces on the domain of size n follow the dynamics of a single interface on the domain of size one, copied over n times using even reflections.*

Figure 2(b) illustrates this conclusion. At the start, the four interfaces are unevenly distributed. However they synchronize after a transient period, forming two phase-locked "breathers". In the case of two interfaces, this result was already obtained in [23].



(a) (b) (c)

FIGURE 2. Simulation of the cubic model (3) with initial conditions consisting of four interfaces. Parameters are $\varepsilon = 0.01$, $D = 50$, $\beta = 0$ and with τ_0 as indicated. The red lines show the locations of interfaces corresponding to the roots of $u = 0$. The areas denoted by + and - correspond to the regions where $u \approx \pm 1$, respectively. The dotted lines correspond to the envelope computation of the interface locations as derived in Appendix C. (a) With $\tau_0 = 0.3$, oscillations eventually die out and four interfaces settle to a steady state. (b) With $\tau_0 = 1$, four interfaces eventually oscillate in synchrony. (c) With $\tau_0 = 3$, the oscillations increase until the interfaces collide resulting in the constant solution $u = -1$ thereafter. Asymptotics correctly predict the time of collision.

In short, the behaviour of n interfaces for the two-component near-shadow system (1, 2) on the domain of size n is fully described by a single interface on the domain of size one. In particular, n interfaces are stable if and only if a single interface is. In fact, one of the following three scenarios give a complete list of possible behaviours:

1. Some interfaces eventually cross each other, leading to annihilation of the two interfaces (figure 2(c)).
2. $2K$ interfaces eventually reach a steady state (figure 2(a)).
3. $2K$ interfaces oscillate indefinitely; the long-time dynamics consist of K "breathers" (see figure 2(b)) that oscillate synchronously (in-phase).

Unlike some other literature e.g. [15, 16, 24] which shows the presence of oscillations as a result of a Hopf bifurcation of the ground state, our results are more "global" as they do not rely on linearization around the steady state. For instance, our results hold even far from the Hopf bifurcation point. In particular, it shows that even if both in-phase and out-of-phase modes become linearly unstable, the solution with two or more interfaces will converge to the in-phase mode.

While we do not carry out the linear stability analysis here, such an approach would yield small eigenvalues that undergo a Hopf bifurcation. What our analysis suggests is that even for values of τ well above the Hopf bifurcation (and with D large), the critical eigenvalue is dominated by its imaginary part with a comparably small real part. Its imaginary part then determines the frequency of the oscillations whereas its real part is proportional to the constant in front of the linear term in envelope equation (4).

In Appendix C we derive the equations of motion of n interfaces. These equations are used to plot the asymptotic prediction in Figure 2, in excellent agreement with the full numerics. However these equations are extremely long and are not used at all in the proof of the main result.

In the derivation of the main result we will assume periodic boundary conditions. The periodicity requires an even number of interfaces, $n = 2K$, so that the value of u to the left of the leftmost interface is the same as its value to the right of the rightmost interface. However the result also holds for Neumann boundary conditions (in which case one can have an odd number of interfaces). This can be seen as follows. Take the system that has n interfaces on a bounded interval with Neumann boundary conditions. Extend the solution using even reflection in space about either endpoint. Then the resulting system on the domain twice the size satisfies periodic boundary conditions and has $2n$ interfaces. We can then apply our results to the extended system to show that the $2n$ interfaces behave just like a single interface in the long-run. Therefore the same is true for the original system of n interfaces.

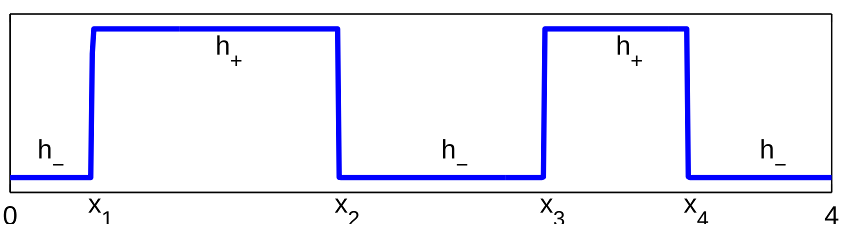
Following the paper [23], the first step in showing the main result is to reduce the motion of n interfaces to the following diffusion-type equation with n moving boundaries:

$$W_{xx} = \varepsilon(W_t + h(x)) - \sigma(x)\varepsilon^2 W \tag{5a}$$

$$\frac{d}{dt}x_i = (-1)^i W(x_i), \quad i = 1 \dots n, \quad n \text{ even} \tag{5b}$$

$$W \text{ is periodic on the domain } [0, n]. \tag{5c}$$

where functions $h(x)$ and $\sigma(x)$ are piecewise constant, alternating between two values with jumps precisely at the interface locations x_j as illustrated here for h:



that is

$$h = \begin{cases} h_- \text{ for } x \in (0, x_1) \cup (x_2, x_3) \cup \dots \cup (x_{2K-2}, x_{2K-1}) \cup (x_{2K}, n) \\ h_+ \text{ for } x \in (x_1, x_2) \cup (x_3, x_4) \dots \cup (x_{2K-1}, x_{2K}) \end{cases}$$

and similarly for σ. This reduction was done in [23] and is summarized in Appendix A for completeness. Note that the ε and t in (5) are not the same as the ε and t in (1); see Appendix A and in particular equation (47) for their values in terms of the original parameters of the system. The n moving boundaries x_i represent interface locations ordered in increasing order. For the basic model (3), these functions are given by

$$h_{\pm} = \frac{3}{2}\tau_0 (\pm 1 + \beta); \qquad \sigma_+ = \sigma_- = -\frac{1}{4};$$

more generally these functions are given in Appendix A.

In [15], the authors took a similar approach of reducing the dynamics of a single interface to a free-boundary problem similar to (5). By solving this problem numerically (assuming piecewise-linear nonlinearities f, g, but without assuming that D is large), they also found that it can capture oscillatory dynamics of the interfaces. The authors also computed the eigenvalues of the associated linear problem analytically, showing directly the existence of the Hopf bifurcation. In a related work [16] the authors studied the motion of two interfaces for piecewise-linear nonlinearities f, g and without assiming large D. They showed the possibility of in-sync oscillations when the interfaces are very close to each other.

We now state the Principal Result for the system (5), which is the more precise statement of the Main Result.

Principal Result 1.1. Consider the system (5) with $n = 2K$ interfaces and with $\varepsilon \ll 1$. Suppose that the interface locations x_i never cross each-other. Then in the limit $t \gg 1$, we have:

$$\begin{aligned} x_{2j-1} &\sim -A\cos(\omega(t+\phi)) + (2j-2) - l_0 + x_0, \quad j = 1 \ldots K \\ x_{2j} &\sim A\cos(\omega(t+\phi)) + (2j-2) + l_0 + x_0, \quad j = 1 \ldots K \end{aligned}$$

where:

$$l_0 = -h_-/(h_+ - h_-)$$

and A is the stable equilibrium of the ODE

$$\frac{dA}{ds} = -\frac{A^3}{4} - \left(\frac{1}{6} - l_0 + l_0^2 + \frac{\sigma_- + (\sigma_+ - \sigma_-)\, l_0}{2}\right) A. \tag{6}$$

The proof of Principal Result 1.1 is given in the following section. It depends on lemmas 2.1 and 2.3 whose proofs are also given there, as well as on lemma 2.2 previously proven in [23] and included here in appendix B for completeness.

2. **Proof of Principal Result 1.1.**

The proof of Principal Result 1.1 is a direct consequence of the following two lemmas.

Lemma 2.1. Consider the system (5). Suppose that the interface locations $x_j(t)$ never collide. Suppose that they are ordered in the increasing order $0 < x_1 < x_2 < \ldots < x_{2K} < 2K$. Define

$$m_1 = \frac{x_1 + x_2}{2}, \quad m_2 = \frac{x_3 + x_4}{2}, \ \ldots, \ \ m_K = \frac{x_{2K-1} + x_{2K}}{2}$$

and similarly, define

$$\hat{m}_1 = \frac{x_2 + x_3}{2}, \quad \hat{m}_2 = \frac{x_4 + x_5}{2}, \ \ldots \ \hat{m}_K = \frac{x_{2K} + (x_1 + 2K)}{2}$$

Then in the limit $t \to \infty$, we have the following properties:

$$m_{j+1} - m_j \to 2 \text{ as } t \to \infty, \quad j = 1 \ldots K-1 \tag{7}$$

$$\hat{m}_{j+1} - \hat{m}_j \to 2 \text{ as } t \to \infty, \quad j = 1 \ldots K-1. \tag{8}$$

Lemma 2.2. Consider the system

$$W_{xx} = \varepsilon(w_t + h(x;\xi)) - \varepsilon^2\sigma(x;\xi)w \tag{9a}$$

$$\frac{d}{dt}\xi(t) = W(\xi, t) \tag{9b}$$

$$W_x = 0 \text{ at } x = 0, \text{ and } x = 1 \tag{9c}$$

$$h(x;\xi) = \begin{cases} h_+ \text{ if } 0 < x < \xi \\ h_- \text{ if } \xi < x < 1 \end{cases} ; \quad \sigma(x;\xi) = \begin{cases} \sigma_+ \text{ if } 0 < x < \xi \\ \sigma_- \text{ if } \xi < x < 1 \end{cases} . \tag{9d}$$

In the limit $\varepsilon \to 0$, the system (5) has solution of the form

$$\xi(t) \sim A(s)\cos(\omega t + \phi_0) + l_0$$

where

$$s = \varepsilon t$$
$$\omega = \sqrt{h_+ - h_-};$$
$$l_0 = -h_-/(h_+ - h_-)$$

and A satisfies

$$A_s = -\frac{A^3}{4} - \left(\frac{1}{6} - l_0 + l_0^2 + \frac{\sigma_- + (\sigma_+ - \sigma_-)\, l_0}{2}\right) A. \tag{10}$$

Lemma 2.2 was proven in [23]; for completeness and reader's convenience, we include a slightly different derivation in appendix A. We first prove Principal Result 1.1 followed by the proof of Lemma 2.1.

Proof of Principal Result 1.1. For reader's convenience, we will give the proof for four interfaces here; the proof is similar for $2K$ interfaces.

By lemma 2.1, there exist constants c_0 and c_1 such that for large t we have:

$$x_1(t) + x_2(t) \sim 2c_0 + 2 \tag{11}$$
$$x_2(t) + x_3(t) \sim 2c_1 + 4 \tag{12}$$
$$x_3(t) + x_4(t) \sim 2c_0 + 6 \tag{13}$$
$$x_4(t) + x_1(t) \sim 2c_1 + 4. \tag{14}$$

Taking (11)-(12)+(13)-(14) we obtain

$$c_0 = c_1.$$

By shifting x_j we may assume without loss of generality that $c_0 = c_1 = 0$. The resulting linear system (11-14) has a one-dimensional null space and its solution is parameterized by a free parameter ξ and may be written as

$$x_1(t) \sim 1 - \xi(t), \quad x_2(t) \sim 1 + \xi(t),$$
$$x_3(t) \sim 3 - \xi(t), \quad x_4(t) \sim 3 + \xi(t).$$

This implies that the solution is an even periodic extension of the interval of size one copied over n times; in particular u is even around $x = 1, 2, 3, \dots n-1$. Therefore the solution in this regime is equivalent to an oscillation of a single interface on domain of size one with Neumann boundary conditions. This is precisely the situation captured by lemma 2.2, which concludes the proof. ■

Proof of Lemma 2.1. Define

$$l_i = \frac{x_{2i} - x_{2i-1}}{2}, \; i = 1 \dots K \tag{15}$$

and define a symmetric unit box of size $2l$ to be

$$B(x;l) = \frac{1}{2}\left(H(x-l) + H(l-x)\right) = \begin{cases} 1 \text{ if } |x| < l \\ 0 \text{ if } |x| > l \end{cases}$$

(where H is the Heaviside function) so that we may write $h(x)$ as

$$h(x) = c + d\sum_{j=1}^{n/2} B(x - m_j, l_j), \qquad \text{where } d = h_+ - h_-, \quad c = h_-. \tag{16}$$

We perform multiple scales analysis. Introduce a slow-time scale

$$s = \varepsilon t, \quad w = W(x,t,s), \quad \xi_j = \xi_j(t,s)$$

and expand

$$W(x,t) = W_0 + W_1\varepsilon + W_2\varepsilon^2 \dots$$

and

$$x_j = \xi_j + \varepsilon\eta_j + \dots$$

Expanding h to two orders, we have

$$h(x) = h_0(x) + \varepsilon h_1(x)$$

where h_0 is as given by (16) but with ξ_j replaced by ξ_{j0}, and

$$h_1(x) = \sum (-1)^j \, d\eta_j \delta(x - \xi_j)$$

where δ is the delta function. We therefore obtain

$$W_{0xx} = 0 \tag{17}$$
$$W_{1xx} = W_{0t} + h_0(x) \tag{18}$$
$$W_{2xx} = W_{1t} + W_{0s} + h_1(x) - \sigma W_0 \tag{19}$$

Similarly, expanding (5c) we obtain

$$\xi_{jt} = (-1)^j \, W_0(\xi_j, t, s) \tag{20}$$
$$\eta_{jt} + \xi_{js} = (-1)^j \, \eta_j W_{0x}(\xi_j, t, s) + (-1)^j \, W_1(\xi_j, t, s). \tag{21}$$

From (17) and periodicity, W_0 is independent of x :

$$W_0(x,t,s) = W_0(t,s)$$

so that (20, 21) becomes

$$\xi_{jt} = (-1)^j \, W_0(t,s); \qquad \eta_{jt} + \xi_{js} = (-1)^j \, W_1(\xi_j, t, s). \tag{22}$$

Integrating (51) assuming periodic b.c. we get

$$W_{0t} = -\frac{1}{n}\int_0^n h_0(x)dx = -c - \frac{d}{n}\sum_{j=1}^{n} (-1)^j \, \xi_j. \tag{23}$$

This suggests that we define a new variable

$$Y = c + \frac{d}{L}\sum_{j=1}^{n} (-1)^j \, \xi_j. \tag{24}$$

From (22) and (23) we then obtain

$$Y_t = \omega^2 W_0; \quad W_{0t} = -Y \qquad \text{where } \omega := \sqrt{d}. \tag{25}$$

From this we obtain

$$W_0 = A(s)\,\omega\cos(\omega t + \Phi(s))\,; \qquad Y = A(s)\,\omega^2 \sin(\omega t + \Phi(s)) \tag{26}$$

We can therefore write

$$\xi_j = (-1)^j\, A(s)\sin(\omega t + \Phi(s)) + B_j(s) \tag{27}$$

with an additional algebraic constraint

$$0 = c + \frac{d}{L}\sum_{j=1}^{n} (-1)^j\, B_j. \tag{28}$$

Next let's compute W_1. We have

$$W_{1xx} = h_0 - \frac{1}{L}\int_0^L h_0, \qquad W_1 \text{ periodic on } [0, L] \tag{29}$$

Define $\hat{F}(x; l, L)$ to be the unique periodic solution to the equation

$$\hat{F}_{xx} = B(x,l) - \frac{1}{L}\int_{-L/2}^{L/2} B(x,l)dx \quad \text{inside } [-L/2, L/2];$$

$$\int_{-L/2}^{L/2} \hat{F}(x)dx = 0 \ \text{ and } \ \hat{F} \text{ is periodic on } [-L/2, L/2].$$

Direct computations show that

$$\hat{F}(x; l, L) = \begin{cases} -\frac{1}{6}\frac{l}{L}(L-l)(L-2l) + (\frac{1}{2} - \frac{l}{L})x^2, & |x| < l \\ -\frac{1}{6}\frac{l}{L}(L-l)(L-2l) + \left(-\frac{l}{L}x^2 + l\,|x| - \frac{l^2}{2}\right), & l < |x| < \frac{L}{2} \\ \hat{F}(\operatorname{mod}(x + \frac{L}{2}, L) - \frac{L}{2}, l, L) & \text{otherwise} \end{cases}. \tag{30}$$

Next define

$$F(x; \xi_1 \ldots \xi_n) = \sum_{j=1}^{n/2} \hat{F}(x - m_j, l_j, L) \tag{31}$$

so that

$$W_1 = dF(x) + R(t,s) \tag{32}$$

and

$$\eta_{jt} + \xi_{js} = (-1)^j\,(dF(x) + R(t,s))\,. \tag{33}$$

First, assume $n = 4$.. We expand (with a slight abuse of notation)

$$m_j = m_j + \varepsilon M_j + \ldots$$
$$l_j = l_j + \varepsilon \Lambda_j + \ldots$$

Then

$$m_{1s} + M_{1t} = \frac{d}{2}\left[F(\xi_2) - F(\xi_1)\right];$$
$$m_{2s} + M_{2t} = \frac{d}{2}\left[F(\xi_4) - F(\xi_3)\right].$$

We now explicitly compute these expressions. Recall that

$$\xi_1 = m_1 - l_1; \qquad \xi_2 = m_1 + l_1$$
$$\xi_3 = m_2 + l_2; \qquad \xi_4 = m_2 - l_2$$

and we compute

$$\begin{aligned} F(\xi_1) &= \hat{F}\left(-l_1, l_1, 4\right) + \hat{F}\left(m_1 - m_2 - l_1, l_2, 4\right) \\ &= \hat{F}\left(l_1, l_1, 4\right) + \hat{F}\left(m_2 - m_1 + l_1, l_2, 4\right) \\ F(\xi_2) &= \hat{F}\left(-l_1, l_1, 4\right) + \hat{F}\left(m_1 - m_2 + l_1, l_2, 4\right) \\ &= \hat{F}\left(l_1, l_1, 4\right) + \hat{F}\left(m_2 - m_1 - l_1, l_2, 4\right) \end{aligned}$$

From (30), note that

$$\hat{F}(x-a,l,L) - \hat{F}(x+a,l,L) = la\left(2 - \frac{4}{L}x\right) \quad \text{provided that } l < x \pm a < L,$$

from which it follows that

$$F(\xi_2) - F(\xi_1) = l_2 l_1 \left(m_2 - m_1 - 2\right).$$

Similarly,

$$\begin{aligned} F\left(\xi_3\right) &= \hat{F}\left(m_2 - m_1 - l_2, l_1, 4\right) + \hat{F}(l_2, l_2, 4) \\ F\left(\xi_4\right) &= \hat{F}\left(m_2 - m_1 + l_2, l_1, 4\right) + \hat{F}(l_2, l_2, 4) \end{aligned}$$

so that

$$F(\xi_4) - F(\xi_3) = l_1 l_2 \left(2 - (m_2 - m_1)\right).$$

Change variables,

$$m_1 = 1 + y_1; \qquad m_2 = 3 + y_2$$

so that

$$\begin{aligned} y_{1s} + M'_{1t} &= \frac{d}{2} l_2 l_1 \left(y_2 - y_1\right); \\ y_{2s} + M'_{2t} &= \frac{d}{2} l_1 l_2 \left(y_1 - y_2\right). \end{aligned}$$

More generally, for $n = 2K$ interfaces, we change variables $m_j = 2j - 1 + y_j, \;\; j = 1 \dots K$ to obtain:

$$\begin{pmatrix} y_{1s} + M_{1t} \\ y_{2s} + M_{2t} \\ \dots \\ y_{Ks} + M_{Kt} \end{pmatrix} = \frac{d}{K} M \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_K \end{pmatrix},$$

$$M = \begin{pmatrix} -\left(l_1 l_2 + l_1 l_3 + \dots l_1 l_K\right) & l_1 l_2 & \dots & l_1 l_K \\ l_2 l_1 & -\left(l_2 l_1 + l_2 l_3 + \dots l_1 l_K\right) & \dots & l_2 l_K \\ \vdots & & & \vdots \\ l_K l_1 & \dots & l_K l_{K-1} & -\left(l_K l_1 + l_K l_3 + \dots l_K l_{K-1}\right) \end{pmatrix}.$$

Integrating each equation from 0 to $2\pi/\omega$ this yields

$$\begin{pmatrix} y_{1s} \\ y_{2s} \\ \dots \\ y_{Ks} \end{pmatrix} = \tilde{M}(s) \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_K \end{pmatrix}$$

where $\tilde{M}_{ij}(s) = \frac{1}{2\pi/\omega} \frac{d}{K} \int_0^{2\pi/\omega} M_{ij}(s,t)dt$. Note that assuming $l_j > 0$ for $j = 1 \dots K$, each off-diagonal entry of $\tilde{M}$ is positive. It follows by Lemma 2.3 below that $y_j \to \bar{y}$ as $t \to \infty$, where $\bar{y} = \frac{1}{K} \sum_j^K y_j$. This shows the formula (7) of lemma 2.1. To show formula (8), apply exactly the same argument after shifting all of the indices by one (so that x_1 becomes x_2, x_2 becomes x_3 ... and x_n becomes x_1). ■

It remains to show the following lemma.

Lemma 2.3. Let $A(t)$ be an $K \times K$ matrix with the following properties:

(a) There exists a constant m such that $A_{ij}(t) \geq m$ for all $t \geq 0$ and for all off-diagonal entries $i \neq j$;
(b) A is symmetric;
(c) $A_{ii}(t) = -\sum_{j \neq i} A_{ij}(t)$.

Suppose that y solves

$$y'(t) = A(t)y(t). \tag{34}$$

Then $y(t) \to \bar{y}\mathbf{1}$ as $t \to \infty$, where $\bar{y}$ is the average $\bar{y} = \frac{1}{K}\sum_{j=1}^{K} y_j(0)$, and $\mathbf{1} = (1, 1, \dots 1)^T$. More precisely, there there exists a constant C such that

$$|y(t) - \bar{y}\mathbf{1}| \leq Ce^{-mKt} \text{ for all } t \geq 0 \tag{35}$$

Proof. First, note that that $y(t) = c\mathbf{1}$ is a solution to (34) for any constant c, since A admits an eigenvalue of zero whose corresponding eigenvector is $\mathbf{1}$. Let $z(t) = y(t) - \bar{y}\mathbf{1}$ where $\bar{y} = \frac{1}{K}\sum_{j=1}^{K} y_j(0)$. Then z also satisfies $z' = Az$ with $\sum z_i = 0$. Multiply both sides by z^T on the left to obtain:

$$\left(|z|^2\right)' = 2z^T Az.$$

Next we claim that $z^T Az \leq -mK\,|z|^2$. To see this we have

$$\begin{aligned}
z^T Az &= -\frac{1}{2}\sum_i \sum_j A_{ij}\,(z_j - z_i)^2 \\
&\leq -\frac{m}{2}\sum_i \sum_j (z_j - z_i)^2 \\
&= -\frac{m}{2}\sum_i \sum_j \left(z_j^2 + z_i^2 - 2z_i z_j\right) \\
&= -\frac{m}{2}\left[2K\sum_i z_i^2 - 2\sum_i z_i\left(\sum_j z_j\right)\right] \\
&= -mK\,|z|^2
\end{aligned}$$

so that $\left(|z|^2\right)' \leq -2mK\,|z|^2$. By Gronwall's inequality, it follows that $|z|^2 \leq Ce^{-2mKt}$ which shows (35). ■

3. **Discussion.** In this paper we established that the behaviour of n interfaces for the two-component near-shadow system (1, 2) on the domain of size n is fully described by a single interface on the domain of size one. In particular, n interfaces are stable if and only if a single interface is. In fact, one of the following three scenarios give a complete list of possible behaviours:

1. Some interfaces eventually cross each other, leading to annihilation of the two interfaces (figure 2(c)).
2. $2K$ interfaces eventually reach a steady state (figure 2(a)).
3. $2K$ interfaces oscillate indefinitely; the long-time dynamics consist of K "breathers" (see figure 2(b)) that oscillate synchronously (in-phase).

Unlike some other literature e.g. [16, 24] which shows the presence of oscillations as a result of a Hopf bifurcation of the ground state, our results are more "global" as they do not rely on linearization around the steady state. For instance, our results hold even far from the Hopf bifurcation point. In particular, it shows that even if both in-phase and out-of-phase modes become linearly unstable, the solution with two or more interfaces will converge to the in-phase mode.

The situation can much richer if D is not assumed to be large [16], or for systems consisting of more than two equations, such as two competing species mediated by a predator [25], or a system with one activator and two inhibitors [26]. In [16], both in-phase as well as out-of-phase oscillations of two interfaces were observed and analysed for a two-component system with piece-wise linear nonlinearities, and with $D = O(1)$. Out-of-phase oscillations were found when the two interfaces were "close" to each-other. It is an interesting question whether a reduced PDE-ODE type system can be derived in the regime where $D = O(1)$, and whether such system can capture the out-of-phase oscillations (or even more exotic dynamics) that are not present when D is large.

In [26], for a three-component system, in addition to synchronous and asynchronous oscillations, the authors found regimes where chaotic oscillations of two interfaces was observed. An interesting open question is whether the multiple-scales type methods can be applied to the three-component system in some sub-regime, and if so, what kind of envelope equations can be derived from it.

4. Appendix A: Moving boundary problem.

For completeness, we include the derivation of the moving boundary problem from [23]. First, suppose that there exists a constant w_0 and two constants $u_+ \neq u_-$ such that boundary value problem

$$U_{0yy} + f(U_0, w_0) = 0 \tag{36}$$

$$U_0(y) \to u_\pm \text{ as } y \to \mp\infty \tag{37}$$

admits a solution. Then $U_0(-y)$ also solves (36). (for the special case of f as in (3), we may choose $w_0 = 0$, $u_\pm = \pm 1$, $U_0(y) = -\tanh(y)$). The required conditions are that $u_\pm$ and w_0 must satisfy

$$\int_{u_-}^{u_+} f(u, w_0)du = 0, \quad f(u_+, w_0) = 0 = f(u_-, w_0), \tag{38}$$

with $u_+ \neq u_-$. We further expand

$$u = u_0 + \frac{1}{D}u_1 + \dots, \quad w = w_0 + \frac{1}{D}w_1 + \dots .$$

to obtain

$$0 = \varepsilon^2 u_{0xx} + f(u_0, w_0), \tag{39}$$

$$Du_{0t} = \varepsilon^2 u_{1xx} + f_u(u_0, w_0)u_1 + f_w(u_0, w_0)w_1. \tag{40}$$

In the subsequent analysis, the time scaling will be chosen in such a way that the term Du_{0t} is of the same order as the other terms in (39). Consider a single interface located at $x = \xi(t)$ in the domain $[0, 1]$, with $u \sim u_+$ for $0 < x < \xi$ and with $u \sim u_-$ for $\xi < x < 1$. In the inner region we have

$$u_0(x, t) = U_0\left(\frac{x - \xi(t)}{\varepsilon}\right) = U_0(y) \tag{41}$$

where U_0 is defined (36). Multiplying (39) by u_{0_x} and integrating by parts over the domain, we obtain

$$-\xi'(t)\int_0^1 u_{0x}^2 dx = \frac{1}{D}\int_0^1 f_w w_1 u_{0x} dx. \tag{42}$$

where we neglected the exponentially small boundary terms. In the inner variables, we approximate $w_1 \sim w_1(\xi)$. Rearranging, we now have an equation for the dynamics of the interface

$$\xi_t = \frac{\varepsilon}{D}\frac{\int_{u_-}^{u_+} f_w du}{\int_{-\infty}^{\infty} U_{0y}^2 dy} w_1(\xi). \tag{43}$$

Expanding in $\frac{1}{D}$, from the equation for w in (1), we obtain

$$\frac{\tau}{\epsilon} w_{1t} = w_{1xx} + g(u_0, w_0) + \frac{1}{D} g_u(u_0, w_0) u_1 + \frac{1}{D} g_w(u_0, w_0) w_1 \tag{44}$$

Note that we also kept to $O(1/D)$ terms here. These are not needed to compute the Hopf bifurcation threshold but are necessary for envelope calculation. Away from the interface, we neglect the diffusion term u_{1xx} as well the left hand side in (40), so that

$$u_1 \sim -\frac{f_w(u_0, w_0)}{f_u(u_0, w_0)} w_1.$$

and we obtain a moving boundary problem

$$\frac{\tau}{\epsilon} w_{1t} = w_{1xx} + g(x) + \frac{1}{D}\sigma(x) w_1 \tag{45}$$

where

$$g(x) = \begin{cases} g(u_+, w_0), & x < \xi \\ g(u_-, w_0), & x > \xi \end{cases},$$

$$\sigma(x) = \begin{cases} \left(g_w - \frac{f_w}{f_u} g_u\right)_{u=u_+, w=w_0}, & x < \xi \\ \left(g_w - \frac{f_w}{f_u} g_u\right)_{u=u_+, w=w_0}, & x > \xi \end{cases}.$$

and with ξ controlled by (43). A-posteriori analysis suggests the following rescaling:

$$\tau = \tau_0 \frac{D}{\varepsilon}$$

$$w_1 = \sqrt{D}\frac{\int_{-\infty}^{\infty} U_{0y}^2 dy}{\int_{u_-}^{u_+} f_w du}\tau_0 W$$

$$t = \frac{\tau_0\sqrt{D}}{\varepsilon}\hat{t}$$

which yields the scaled system

$$W_{xx} = \tilde{\varepsilon}(W_{\tilde{t}} + h(x)) - \sigma(x)\tilde{\varepsilon}^2 W, \qquad \frac{d}{d\tilde{t}}\xi = W(\xi, \tilde{t}) \tag{46}$$

where

$$\tilde{\varepsilon} = \frac{1}{\sqrt{D}}, \quad h(x) = -g(x)\frac{\int_{u_-}^{u_+} f_w du}{\int_{-\infty}^{\infty} U_{0y}^2 dy}\tau_0. \tag{47}$$

Dropping the tilde yields the system (9). The reduced system (5) for multiple interfaces is derived in an identical manner.

5. **Appendix B: Single interface.** In this appendix we re-derive the dynamics of a single interface for the problem (5) with Neumann boundary conditions on $[0, 1]$. This was derived in [23], and we include a slightly different self-contained derivation here for completeness. That is, we consider the following problem. . Introduce a

$$W_{xx} = \varepsilon(W_t + h(x)) - \sigma(x)\varepsilon^2 W, \ x \in (0, 1) \tag{48a}$$

$$\frac{d}{dt}\xi = W(\xi, t) \tag{48b}$$

$$W_x = 0 \text{ at } x = 0, 1. \tag{48c}$$

where $h(x)$ can be written in terms of the Heaviside function $H(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ as

$$h(x) = c + dH(\xi - x), \qquad d = h_+ - h_-, \quad c = h_-. \tag{48d}$$

and where

$$\sigma = \begin{cases} \sigma_+ \text{ if } x < \xi \\ \sigma_- \text{ if } x > \xi \end{cases}. \tag{48e}$$

We perform multiple scale analysis on (48). Introduce a slow-time scale

$$s = \varepsilon t, \quad w = W(x, t, s), \quad \xi = \xi(t, s)$$

Expand

$$W = W_0 + W_1\varepsilon + W_2\varepsilon^2 \dots \tag{49}$$

First, we expand

$$H(\xi - x) = H(\xi_0 + \varepsilon\xi_1 - x) = H(\xi_0 - x) + \varepsilon\xi_1\delta(\xi_0 - x) + O(\varepsilon^2)$$

where δ is the delta function. We therefore obtain

$$W_{0xx} = 0 \tag{50}$$

$$W_{1xx} = W_{0t} + c + dH(\xi_0 - x) \tag{51}$$

$$W_{2xx} = W_{1t} + W_{0s} + \xi_1 d\delta(x_0 - x) - \sigma W_0 \tag{52}$$

Similarly, expanding (48b) we obtain

$$\xi_{0t} = W_0(\xi_0, t, s) \tag{53}$$

$$\xi_{1t} + \xi_{0s} = \xi_1 W_{0x}(\xi_0, t, s) + W_1(\xi_0, t, s) \tag{54}$$

Equation (50) along with the boundary conditions $W_{0x} = 0$ at $x = 0, 1$ yields

$$W_0(x, t, s) = W_0(t, s).$$

Integrating equation (51) and using Neumann boundary conditions yields

$$W_{0t} + c + d\xi_0 = 0$$

so that the leading-order behaviour is

$$W_{0t} = -c - d\xi_0; \quad \xi_{0t} = W_0 \tag{55}$$

which is a harmonic oscillator assuming $d > 0$, whose solution is given by

$$\xi_0 = l_0 + A(s)\sin(\omega t + \Phi),$$
$$W_0 = \omega A(s)\cos(\omega t + \Phi)$$

where we defined

$$l_0 := -c/d; \quad \omega^2 := d.$$

Substituting (55) into (51), W_1 satisfies

$$W_{1xx} = d\left(H(\xi_0 - x) - \xi_0\right).$$

The solution to W_1 is then given by

$$W_1 = dF(x) + V(t) \tag{56}$$

where F is the solution to

$$\begin{cases} F_{xx} = H(\xi_0 - x) - \xi_0, \\ F_x(0) = 0 = F_x(1), \\ \int_0^1 F(x)dx = 0. \end{cases}$$

Explicitly we obtain

$$F = \begin{cases} -\frac{\xi_0(x-1)^2}{2} + \frac{\xi_0 - \xi_0^2}{2} + A, & x > \xi_0 \\ \frac{(1-\xi_0)}{2}x^2 + A, & x < \xi_0 \end{cases}, \quad \text{where } A := \frac{\xi_0(2-\xi_0)(\xi_0 - 1)}{6}$$

and evaluating at $x = \xi_0$ we have

$$F(\xi_0) = \frac{1}{3}\xi_0\left(2\xi_0 - 1\right)\left(1 - \xi_0\right). \tag{57}$$

Substituting (56 57), into (52) and integrating for $x = 0 \ldots 1$ we obtain

$$0 = V_t + W_{0s} + \xi_1 d - \hat{\sigma} W_0 \tag{58}$$

where

$$\hat{\sigma} = \left(\sigma_+ - \sigma_-\right)\xi_0 + \sigma_- \tag{59}$$

and (54) simplifies to

$$\xi_{1t} + \xi_{0s} = dF(\xi_0) + V. \tag{60}$$

Eliminating V and V_t from (58) and (60) yields

$$\xi_{1tt} + \omega^2 \xi_1 = dF'(\xi_0)\xi_0' - 2W_{0s} + \left\{\left(\sigma_+ - \sigma_-\right)\xi_0 + \sigma_-\right\} W_0$$

where $\omega^2 := d$, so that

$$\xi_{1tt} + \omega^2\xi_1 = d\left(F(\xi_0)\right)_t - \omega\left[\left(2A_s - \left\{\left(\sigma_+ - \sigma_-\right)\xi_0 + \sigma_-\right\} A\right)\cos\left(\omega t + \Phi\right) - 2A\Phi_s \sin\left(\omega t + \Phi\right)\right] \tag{61}$$

Multiplying both sides of (61) by $\sin(\omega t + \Phi)$ and integrating on $t = 0 \ldots 2\pi/\omega$ we obtain $A\Phi_s = 0$. Similarly, multiplying (61) by $\cos\left(\omega t + \Phi\right)$ we obtain

$$\begin{aligned} \int_0^{2\pi/\omega} d\left(F(\xi_0)\right)_t \cos\left(\omega t + \Phi\right) dt &= \omega d \int_0^{2\pi/\omega} F(\xi_0)\sin\left(\omega t + \Phi\right) dt \\ &= -\pi\frac{A^3}{2} - \frac{\pi}{3}\left(1 - 6l_0 + 6l_0^2\right) A \\ &= \pi\left[\left(2A_s - \left(\sigma_- + \left(\sigma_+ - \sigma_-\right) l_0\right) A\right)\right] \end{aligned}$$

so that

$$A_s = -\frac{A^3}{4} - \left(\frac{1}{6} - l_0 + l_0^2 + \frac{\left(\sigma_- + \left(\sigma_+ - \sigma_-\right) l_0\right)}{2}\right) A. \tag{62}$$

6. **Appendix C: dynamics of $2K$ interfaces.** We recall from the proof of the Lemma 2.1, equations (26-27) and (32-33) are

$$W_0 = A(s)\,\omega\cos(\omega t + \Phi(s)) \tag{63}$$

$$Y = A(s)\,\omega^2\sin(\omega t + \Phi(s)) \tag{64}$$

$$\xi_j = (-1)^j A(s)\sin(\omega t + \Phi(s)) + B_j(s) \tag{65}$$

$$W_1 = dF(x) + R(t,s) \tag{66}$$

$$\eta_{jt} + \xi_{js} = (-1)^j \left(dF(\xi_j) + R(t,s)\right) \tag{67}$$

Integrating (19) we then obtain

$$2K\left(R_t + W_{0s}\right) - W_0\int_0^{2K}\sigma(x)dx + \sum (-1)^j d\eta_j = 0. \tag{68}$$

Define

$$Y_1 = \frac{\sum_{j=1}^{2K}(-1)^j\eta_j}{2K}$$

Then Y_1 satisfies

$$Y_{1t} + \frac{\sum(-1)^j\xi_{js}}{2K} = \frac{d\sum_{j=1}^{2K}F(\xi_j)}{2K} + R(s,t) \tag{69}$$

Notice that $Y = c + \frac{d}{2K}\sum(-1)^j\xi_j$, which implies $\frac{\sum(-1)^j\xi_{js}}{2K} = \frac{Y_s}{d}$ Thus

$$Y_{1t} + \frac{Y_s}{d} = \frac{d\sum_{j=1}^{2K}F(\xi_j)}{2K} + R(s,t) \tag{70}$$

and

$$\frac{\int_0^{2K}\sigma(x)dx}{2K} = (\sigma^+ - \sigma^-)\frac{\sum_1^{2K}(-1)^j\xi_j}{2K} + \sigma^- = (\sigma^+ - \sigma^-)\frac{Y-c}{d} + \sigma^- \tag{71}$$

Taking the derivative of (70) and combining it with (68), we obtain

$$Y_{1tt} + \omega^2 Y_1 = \frac{d\sum_{j=1}^{2K}F_x(\xi_j)\xi_{jt}}{2K} - \frac{Y_{st}}{d} - W_{0s} + \frac{W_0\int_0^{2K}\sigma(x)dx}{2K}. \tag{72}$$

From (67) we have

$$R(t,s) = Y_{1t} + \frac{Y_s}{d} - \frac{d\sum_{j=1}^{2K}F(\xi_j)}{2K}.$$

Plugging it back into (67),we finally get

$$\eta_{jt} + \xi_{js} = (-1)^j\left(dF(\xi_j) + Y_{1t} + \frac{Y_s}{d} - \frac{d\sum_{j=1}^{2K}F(\xi_j)}{2K}\right). \tag{73}$$

Multiplying both sides of (72) by $\sin(\omega t + \Phi)$ and integrating on $t = 0\ldots 2\pi/\omega$ we obtain $A\Phi_s = 0$. Similarly, multiplying (72) by $\cos(\omega t + \Phi)$ we obtain

$$\int_0^{2\pi/\omega}\left(\frac{d\sum_{j=1}^{2K}F_x(\xi_j)\xi_{jt}}{2K} - \frac{Y_{st}}{d} - W_{0s} + \frac{W_0\int_0^{2K}\sigma(x)dx}{2K}\right)\cos(\omega t + \Phi) = 0 \tag{74}$$

Integrating (73) on $t = 0\ldots 2\pi/\omega$ we obtain

$$B_{js} = (-1)^j \int_0^{2\pi/\omega} \left(dF(\xi_j) + \frac{Y_s}{d} - \frac{d \sum_{j=1}^{2K} F(\xi_j)}{2K} \right) \tag{75}$$

This yields the reduced ODE systems to describe the evolution of the envelope:

$$\begin{cases} W_0 = A\left(s\right)\omega\cos\left(\omega t + \Phi\left(s\right)\right) \\ Y = A\left(s\right)\omega^2 \sin(\omega t + \Phi\left(s\right)) \\ \xi_j = \left(-1\right)^j A(s)\sin\left(\omega t + \Phi(s)\right) + B_j(s) \\ \int_0^{2\pi/\omega} \left(\frac{d\sum_{j=1}^{2K} F_x(\xi_j)\xi_{jt}}{2K} - \frac{Y_{st}}{d} - W_{0s} + \left(\frac{(\sigma^+ - \sigma^-)(Y-c)}{d} + \sigma^-\right)W_0 \right)\cos\left(\omega t + \Phi\right)dt = 0 \\ B_{js} = (-1)^j \int_0^{2\pi/\omega} \left(dF(\xi_j) + \frac{Y_s}{d} - \frac{d\sum_{j=1}^{2K} F(\xi_j)}{2K} \right) \\ 0 = c + \frac{d}{2K}\sum_{j=1}^{2K} \left(-1\right)^j B_j \end{cases} \tag{76}$$

We used Maple to evaluate these integrals for the particular case of four interfaces, resulting in a set of ODE's for B_j and A. The right-hand side of each of these equations involves a cubic polynomial in B_j and A with over 40 terms. These ODE's were then integrated numerically. The resulting numerical solution is shown using dotted lines in Figure 2.

REFERENCES

[1] Hardway, H & Li, Y.-X. (2010) Stationary and oscillatory fronts in a two-component genetic regulatory network model. *Physica D* **239**, 1650–1661.

[2] Li, Y.-X. (2003) Tango waves in a bidomain model of fertilization calcium waves. *Physica D,* **186**, 27–49.

[3] Lengyel, I & Epstein, I. (1991) Modeling of turing structures in the chlorite-iodide-malonic acid-starch reaction system. *Science, Vol* **251**, 650–652.

[4] Vanag, V. K & Epstein, I. R. (2007) Localized patterns in reaction-diffusion systems. *Chaos* **17**.

[5] Kaminaga, A, Vanag, V. K, & Epstein, I. R. (2006) A reaction–diffusion memory device. *Angewandte Chemie* **45**.

[6] Kolokolnikov, T & Tlidi, M. (2007) Spot deformation and replication in the two-dimensional belousov-zhabotinski reaction-diffusion system. *Physical Review Letters, Vol* **98**.

[7] Gurevich, S. V, Amiranashvili, S, & Purwins, H. G. (2006) Breathing dissipative solitons in three component reaction-diffusion system. *Physical Review E* **74**.

[8] van Heijster, P, Doelman, A, & Kaper, T. J. (2008) Pulse dynamics in a three-component system: Stability and bifurcations. *Physica D* **237**, 3335–3368.

[9] Volpert, V & Petrovskii, S. (2009) Reaction–diffusion waves in biology. *Physics of Life Reviews* **6**, 267–310.

[10] Banerjee, M & Petrovski, S. (2011) Self-organised spatial patterns and chaos in a ratio-dependent predator–prey system. *Theor Ecol* **4**, 37–53.

[11] Meron, E, Yizhaq, H, & Gilad, E. (2007) Localized structures in dryland vegetation: forms and functions. *Chaos: An Interdisciplinary Journal of Nonlinear Science* **17**, 037109.

[12] Koga, S & Kuramoto, Y. (1980) Localized patterns in reaction-diffusion systems. *Progress of Theoretical Physics* **63**, 106–121.

[13] Hagberg, A & Meron, E. (1994) Pattern formation in non-gradient reaction-diffusion systems: the effect of front bifurcations. *Nonlinearity* **7**, 805–835.

[14] Haim, D, Li, G, Ouyang, Q, McCormick, W. D, Swinney, H. L, Hagberg, A, & Meron, E. (1996) Breathing spots in a reaction-diffusion system. *Phys. Rev. Lett* **77**, 190–193.

[15] Nishiura, Y & Mimura, M. (1989) Layer oscillations in reaction-diffusion systems. *SIAM J.Appl. Math* **49**, 481–514.

[16] Ikeda, T & Nishiura, Y. (1994) Pattern selection for two breathers. *SIAM Journal on Applied Mathematics* **54**, 195–230.

[17] Ikeda, H & Ikeda, T. (2000) Bifurcaiton phenomena from standing pulse solutions in some reaction-diffusion systems. *J. Dyn. Diff. Eqns* **12**.

[18] Ei, S, Ikeda, H, & Kawana, T. (2008) Dynamics of front solutions in a specific reaction-diffusion system in one dimension. *Japan J. Indust. Appl. Math* **25**, 117–147.
[19] Nagayama, M, Ueda, K.-i, & Yadome, M. (2010) Numerical approach to transient dynamics of oscillatory pulses in a bistable reaction–diffusion system. *Japan journal of industrial and applied mathematics* **27**, 295–322.
[20] Muratov, C & Osipov, V. V. (1996) General theory of instabilities for patterns with sharp interfaces in reaction-diffusion systems. *Phys, Rev. E* **53**, 3101–3116.
[21] Ward, M. J & Wei, J. (2003) Hopf bifurcation and oscillatory instabilities of spike solutions for the one-dimensional gierer-meinhardt model. *J. Nonlinear Science* **13**, 209–264.
[22] Chen, W & Ward, M. J. (2009) Oscillatory instabilities and dynamics of multi-spike patterns for the one-dimensional gray-scott model. *European Journal of Applied Mathematics* **20**, 187–214.
[23] McKay, R, Kolokolnikov, T, & Muir, P. (2012) Interface oscillations in reaction-diffusion systems above the hopf bifurcation. *Discrete & Continuous Dynamical Systems-Series B* **17**.
[24] Ohta, T, Ito, A, & Tetsuka, A. (1990) Self-organization in an excitable reaction-diffusion system: synchronization of oscillatory domains in one dimension. *Physical Review A* **42**, 3225.
[25] Ikeda, T & Mimura, M. (1993) An interfacial approach to regional segregation of two competing species mediated by a predator. *Journal of Mathematical Biology* **31**, 215–240.
[26] Suzuki, M, Ohta, T, Mimura, M, & Sakaguchi, H. (1995) Breathing and wiggling motions in three-species laterally inhibitory systems. *Physical Review E* **52**, 3645–3655.

E-mail address: xieshuangquan2013@gmail.com
E-mail address: tkolokol@gmail.com